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Singular self-dual Zollfrei metrics and twistor correspondence

Fuminori Nakata*

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro, Tokyo 153-8914, Japan

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Abstract

We construct examples of singular self-dual Zollfrei metrics explicitly, by patching a pair of Petean's self-dual split-signature metrics. We prove that there is a natural one-to-one correspondence between these singular metrics and a certain set of embeddings of $\mathbb{RP}^3$ to $\mathbb{CP}^3$ which has one singular point. This embedding corresponds to an odd function on $\mathbb{R}$ that is rapidly decreasing and pure imaginary valued. The one-to-one correspondence is explicitly given by using the Radon transform.

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Keywords: Zollfrei metric; Twistor correspondence

1. Introduction

A *Zollfrei metric*, which was introduced by Guillemin [6], is an indefinite metric of a manifold whose maximal null geodesics are all closed. C. LeBrun and L.J. Mason investigated the self-dual Zollfrei metric of signature (2, 2), and constructed its twistor correspondence [11]. They proved that only $S^2 \times S^2$ and $(S^2 \times S^2)/\mathbb{Z}_2$ admit a self-dual Zollfrei conformal structure. Using the twistor correspondence, they also proved that such a structure on $(S^2 \times S^2)/\mathbb{Z}_2$ is rigid, and, in contrast, $S^2 \times S^2$ admits of many such structures. In the case of $S^2 \times S^2$, the corresponding twistor space is given by a pair $(\mathbb{CP}^3, P)$, where P is the image of a totally real embedding $\mathbb{RP}^3 \to \mathbb{CP}^3$. Their theorem says that there is a one-to-one correspondence between self-dual Zollfrei conformal structures on $S^2 \times S^2$ and the pairs $(\mathbb{CP}^3, P)$, at least in the neighborhoods of the standard structures.

On the other hand, Tod [16] and Kamada [9] independently constructed infinitely many examples of S^1-invariant scalar-flat indefinite Kähler metrics on $S^2 \times S^2$, which are automatically self-dual. Because the Zollfrei condition is an open condition in the space of self-dual metrics [11], these examples should contain many self-dual Zollfrei metrics, and a natural problem here is to decide whether or not all of them are Zollfrei. These examples are written explicitly in closed form, so it might be possible to write down explicitly the twistor correspondence for such metrics. We are not going to pursue these questions in this article; instead, we generalize the formulation to admit a certain singularity,

* Tel.: +81 3 5465 7087.
E-mail address: nakata@ms.u-tokyo.ac.jp.

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and we construct explicit examples of the singular self-dual Zollfrei metric, whose twistor correspondence is written down explicitly.

While LeBrun and Mason's theorem stands only in the neighborhood of the standard metric because they are using an inverse function theorem for Banach space, our examples contain many metrics far from the standard one. We use the Radon transform to write down the twistor correspondence for our singular metric.

In [14], J. Petean classified the compact complex surfaces which admit indefinite Kähler–Einstein metrics. Petean constructed many self-dual metrics on $\mathbb{R}^4$ to show that the complex tori or the primary Kodaira surfaces admit many such metrics. Our examples of the singular self-dual Zollfrei metric are constructed by patching a pair of J. Petean's metrics on $\mathbb{R}^4$.

Our main theorem is to establish the twistor correspondence for some class of singular metrics. We construct the singular metrics and the singular twistor spaces respecting a certain fiber bundle structure over S^2 and $\mathbb{CP}^2$, respectively. We prove the main theorem making use of a fiber bundle structure over a natural double fibration $S^2 \leftarrow Z \rightarrow \mathbb{CP}^2$ for some Z, which is the twistor correspondence for the standard Zoll metric on S^2. Note that a Zoll metric on a smooth manifold is a Riemannian metric whose geodesics are all closed, and the simplest one is the standard metric on S^2. The general case of twistor correspondence for Zoll structure is established by LeBrun and Mason [10].

In [4], Dunajski and West investigated the neutral anti-self-dual conformal structures with a null conformal Killing vector field. While they explicitly gave the local classification of such structures using twistor methods, few global examples of such structures are known, and our singular metrics are regarded as such global examples. Note that our singular metric has a natural S^1-action; however, the corresponding Killing vector field of this action is different from the null conformal Killing vector field that gives the above structure. It would be an interesting problem to generalize their classification to the metrics admitting the singularity introduced in this article.

The organization of the paper is as follows: in Section 2, we recall the statement of the twistor correspondence for the Zoll or Zollfrei structure, respectively following LeBrun and Mason [10,11]. We describe the twistor correspondence for the standard structure on S^2 or $S^2 \times S^2$ by introducing local coordinates that we use later. In Section 3, we introduce the definition of the singular self-dual Zollfrei metrics and the singular twistor spaces, and we formulate a conjecture of the twistor correspondence between them (Conjecture 9).

In Section 4, we construct explicit examples of the singular self-dual Zollfrei metric by patching two Petean's metrics. Each example corresponds to an element of the set $\mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ defined below. Let $\mathcal{S}(\mathbb{R}^2)$ be the set of rapidly decreasing real functions on $\mathbb{R}^2$, and $\mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ be the subset of $\mathcal{S}(\mathbb{R}^2)$ consisting of $SO(2)$-invariant elements, which we call axisymmetric functions.

In Sections 5 and 6, we construct explicit examples of a singular twistor space. Each twistor space corresponds to an element of $\mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$, where $\mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$ is the set of odd functions that are rapidly decreasing and pure imaginary valued. Our main theorem (Theorem 24) says that our conjecture holds when we restrict ourselves to the abovementioned examples. This correspondence is given explicitly as a transform between $f(x) \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ and $h(t) \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$, by using the Radon transform. In Section 7 we give the proof of Theorem 24, and the Appendix is the review of the Radon transform.

2. Standard model

Zoll projective structures: A Zoll metric on a smooth manifold M is a Riemannian metric whose geodesics are all closed (cf. [5]). An example of such a metric is the standard metric on S^2. A Zoll projective structure on M is a projectively equivalent class of torsion-free connections on the tangent bundle TM whose geodesics are all closed, where two torsion-free connections are said to be projectively equivalent if and only if they have exactly the same unparameterized geodesics [10]. Each Zoll metric defines a Zoll projective structure.

C. LeBrun and L.J. Mason proved the following property in [10]; there is a natural one-to-one correspondence between

- equivalence classes of Zoll projective structures on S^2,
- equivalence classes of totally real embeddings $\iota : \mathbb{RP}^2 \rightarrow \mathbb{CP}^2$,

in neighborhoods of the standard projective structure on S^2 and the standard embedding of $\mathbb{RP}^2$. We call this correspondence the twistor correspondence for Zoll projective structures. This correspondence is characterized by the following condition; there is a double fibration $S^2 \xleftarrow{\mathfrak{p}} \mathcal{D} \xrightarrow{\mathfrak{q}} \mathbb{CP}^2$ such that

(1) $\mathfrak{q}$ is a continuous surjection and $\mathfrak{p}$ is a complex disk bundle, i.e. for each $x \in S^2$, $\mathcal{D}_x = \mathfrak{p}^{-1}(x)$ is biholomorphic to the complex unit disk,
(2) $\mathfrak{q}_x : \mathcal{D}_x \to \mathbb{CP}^2$ is holomorphic on the interior of $\mathcal{D}_x$, and $\mathfrak{q}_x(\partial\mathcal{D}_x) \subset N$, where $\mathfrak{q}_x$ is the restriction of $\mathfrak{q}$ on $\mathcal{D}_x$ and $N = \iota(\mathbb{RP}^2)$,
(3) the restriction of $\mathfrak{q}$ on $\mathcal{D} - \partial\mathcal{D}$ is bijective onto $\mathbb{CP}^2 - N$,
(4) $\{\mathfrak{p}(\mathfrak{q}^{-1}(y))\}_{y\in N}$ is equal to the set of geodesics on S^2.

The conditions (2) and (3) say that $\{\mathfrak{q}(\mathcal{D}_x)\}_{x\in S^2}$ is a family of holomorphic disks on $\mathbb{CP}^2$ with boundaries on N which foliate $\mathbb{CP}^2 - N$.

For the standard projective structure on S^2, the twistor correspondence is described by the following diagrams which are explained below:

$$\begin{tikzcd} & \mathbb{P}(TS^2) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S^2 & & \mathbb{RP}^2 \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{D}(TS^2) \arrow[dl, "\mathfrak{p}"'] \arrow[dr, "\mathfrak{q}"] & \\ S^2 & & \mathbb{CP}^2 \end{tikzcd} \tag{1}$$

We call the left diagram the real twistor correspondence and the right one the complex twistor correspondence.

We denote $S^2 = \{t \in \mathbb{R}^3 : \|t\|^2 = 1\}$, $TS^2 = \{(t, v) \in S^2 \times \mathbb{R}^3 : \langle t, v\rangle = 0\}$ and $\mathbb{P}(TS^2) = \{(t, [v]) \in S^2 \times \mathbb{RP}^2 : \langle t, v\rangle = 0\}$. Let $p : \mathbb{P}(TS^2) \to S^2$ be the projection and $q(t, [v]) = [t \times v]$, then we have the left diagram in (1).

Here we introduce a local coordinate system. We take an open covering $\{D_+, D_-, W\}$ of S^2 where $D_\pm = \{t \in S^2 : \pm t_3 > 0\}$ and $W = S^2 - \{(0, 0, \pm 1)\}$. We define local coordinates $(x_1^\pm, x_2^\pm) \in \mathbb{R}^2 \simeq D_\pm$ and $(\alpha, \beta) \in \mathbb{R}/2\pi\mathbb{Z} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \simeq W$ by

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \pm\left(1 + (x_1^\pm)^2 + (x_2^\pm)^2\right) \begin{pmatrix} x_1^\pm \\ x_2^\pm \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ \sin\beta \end{pmatrix}.$$

The circle bundle $\mathbb{P}(TS^2)$ is trivial over $D_\pm$ and W, and we define the fiber coordinate $\zeta^\pm$ and ξ by

$$\begin{aligned} D_\pm \times (\mathbb{R} \cup \{\infty\}) \ni \left(x_1^\pm, x_2^\pm, \zeta^\pm\right) &\longleftrightarrow \left[-\zeta^\pm \frac{\partial}{\partial x_1^\pm} + \frac{\partial}{\partial x_2^\pm}\right] \in \mathbb{P}(TS^2)\,|_{D_\pm}, \\ W \times (\mathbb{R} \cup \{\infty\}) \ni (\alpha, \beta, \xi) &\longleftrightarrow \left[-\xi \frac{\partial}{\partial \alpha} + \cos^2\beta \frac{\partial}{\partial \beta}\right] \in \mathbb{P}(TS^2)\,|_W . \end{aligned} \tag{2}$$

The coordinate change is given by

$$\begin{cases} x_1^\pm = \cos\alpha\cot\beta, \\ x_2^\pm = \sin\alpha\cot\beta, \end{cases} \qquad \zeta^\pm = \frac{-\xi\sin\alpha\tan\beta + \cos\alpha}{\xi\cos\alpha\tan\beta - \sin\alpha}. \tag{3}$$

In terms of these coordinates, the map $q : \mathbb{P}(TS^2) \to \mathbb{RP}^2$ is described by

$$\begin{aligned} &\left(x_1^\pm, x_2^\pm, \zeta^\pm\right) \longmapsto \left[1 : \zeta^\pm : -x_1^\pm - x_2^\pm\zeta^\pm\right], \\ &(\alpha, \beta, \xi) \longmapsto [\xi\cos\alpha\tan\beta - \sin\alpha : -\xi\sin\alpha\tan\beta + \cos\alpha : \xi] . \end{aligned} \tag{4}$$

We define a complex disk bundle $\mathbb{D}(TS^2)$ over S^2 as a closure of one of the two connected components of $\mathbb{P}(T_\mathbb{C}S^2) - \mathbb{P}(TS^2)$, where $\mathbb{P}(T_\mathbb{C}S^2)$ is the complex projectivization of $T_\mathbb{C}S^2 = TS^2 \otimes \mathbb{C}$. The choice of the component is not essential because these components are canonically isomorphic by the complex conjugation.

Extending real parameters $\zeta^{\pm}$ and ξ to the complex parameter in the upper or lower half plane $\mathbb{H}_{\pm} = \{z \in \mathbb{C} : \pm \mathrm{Im}\, z \geq 0\}$, we can introduce the trivialization of $\mathbb{D}(TS^2)$ by

$$\mathbb{D}(TS^2)\,|_{D_{\pm}} \simeq D_{\pm} \times (\mathbb{H}_{\pm} \cup \{\infty\}) \ni (x_1^{\pm}, x_2^{\pm}, \zeta^{\pm}),$$
$$\mathbb{D}(TS^2)\,|_{W} \simeq W \times (\mathbb{H}_{+} \cup \{\infty\}) \ni (\alpha, \beta, \xi).$$

The coordinate change is given by the same formulae as (3) with complex coordinates. Then the map $\mathfrak{q} : \mathbb{D}(TS^2) \to \mathbb{CP}^2$ is obtained by the analytic continuation of q, i.e. $\mathfrak{q}$ is given by the same formula as (4). It is easy to check that the above conditions (1), (2), (3) and (4) hold if we put $\mathcal{D} = \mathbb{D}(TS^2)$.

Self-dual Zollfrei metrics: A Zollfrei metric on a smooth manifold M is an indefinite metric whose null geodesics are all closed (cf. [6]). In [11], LeBrun and Mason investigated self-dual Zollfrei neutral metrics on four-dimensional manifolds, where a neutral metric is an indefinite metric with signature (n, n) which is also called an indefinite metric with split signature. An example of such a metric is the standard metric g_0 on $S^2 \times S^2$ given by $g_0 = \pi_1^* h_{S^2} - \pi_2^* h_{S^2}$, where π_i is the projection to the ith S^2 and h_{S^2} is the standard Riemannian metric on S^2.

LeBrun and Mason proved the following property in [11]; let $M = S^2 \times S^2$ and g be a self-dual Zollfrei neutral metric on M, then every β-*surface* on M is homeomorphic to S^2. By definition, β-*plane* is a tangent null 2-plane at a point whose bivector is anti-self-dual, and β-*surface* is a maximal embedded surface whose tangent plane is β-plane at every point.

LeBrun and Mason constructed the twistor correspondence for self-dual Zollfrei metrics. The statement is as follows; there is a natural one-to-one correspondence between

- equivalence classes of self-dual Zollfrei conformal structures on $S^2 \times S^2$,
- equivalence classes of totally real embeddings $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$,

in neighborhoods of the standard conformal structure $[g_0]$ and the standard embedding of $\mathbb{RP}^3$. They also proved that the Zollfrei condition is an open condition in the space of self-dual neutral metrics. It implies that the term 'Zollfrei' is removable in the above statement. This correspondence is characterized by the following condition; there is a double fibration $S^2 \times S^2 \xleftarrow{\wp} \hat{\mathcal{Z}} \xrightarrow{\Psi} \mathbb{CP}^3$ such that

(1) Ψ is a continuous surjection and $\wp$ is a complex disk bundle, i.e. for each $x \in S^2 \times S^2$, $\hat{\mathcal{Z}}_x = \wp^{-1}(x)$ is biholomorphic to the complex unit disk,
(2) $\Psi_x : \hat{\mathcal{Z}}_x \to \mathbb{CP}^3$ is holomorphic on the interior of $\hat{\mathcal{Z}}_x$ and $\Psi_x(\partial \hat{\mathcal{Z}}_x) \subset P$, where Ψ_x is the restriction of Ψ on $\hat{\mathcal{Z}}_x$ and $P = \iota(\mathbb{RP}^3)$,
(3) the restriction of Ψ on $\hat{\mathcal{Z}} - \partial \hat{\mathcal{Z}}$ is bijective onto $\mathbb{CP}^3 - P$,
(4) $\{\wp(\Psi^{-1}(y))\}_{y \in P}$ is equal to the set of β-surfaces on $S^2 \times S^2$.

The conditions (2) and (3) say that $\{\Psi(\hat{\mathcal{Z}}_x)\}_{x \in S^2 \times S^2}$ is a family of holomorphic disks on $\mathbb{CP}^3$ with boundaries on P which foliate $\mathbb{CP}^3 - P$.

The twistor correspondence for the standard metric g_0 on $M = S^2 \times S^2$ is explained in Lemma 8.1 of [11]. For later convenience, we give an alternative description of the double fibration for g_0, and we describe the situation by using local coordinates. The twistor correspondence is described by the following diagrams, which are explained below:

$$\begin{tikzcd}
 & \hat{F} \arrow[dl, "\hat{p}"'] \arrow[drr, "\hat{\Phi}_0"] & & \\
M & F \arrow[u, dotted] \arrow[r] \arrow[dl, "\tilde{p}"'] \arrow[d] \arrow[drr, "\Phi_0"] & L_{\perp} \arrow[dl] \arrow[dr] & \mathbb{RP}^3 \\
TS^2 \arrow[u, dotted] \arrow[d] & \mathbb{P}(TS^2) \arrow[dl, "p"'] \arrow[drr, "q"] & & \mathcal{O}_{\mathbb{R}}(1) \arrow[u, dotted] \arrow[d, "\pi"] \\
S^2 & & & \mathbb{RP}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \hat{\mathcal{Z}} \arrow[dl, "\wp"'] \arrow[dr, "\Psi_0"] & \\
M & \mathcal{Z} \arrow[u, dotted] \arrow[dl, "\tilde{\mathfrak{p}}"'] \arrow[d] \arrow[dr, "\Phi_{\mathbb{C},0}"] & \mathbb{CP}^3 \\
TS^2 \arrow[u, dotted] \arrow[d] & \mathbb{D}(TS^2) \arrow[dl, "\mathfrak{p}"'] \arrow[dr, "\mathfrak{q}"] & \mathcal{O}(1) \arrow[u, dotted] \arrow[d, "\pi"] \\
S^2 & & \mathbb{CP}^2
\end{tikzcd}
\tag{5}$$

Here $p, q, \mathfrak{p}$ and $\mathfrak{q}$ are the same maps as in the diagrams (1), and the upward arrows are inclusions.

It is convenient to use an identification of $S^2 \times S^2$ with $\widetilde{Gr}_2(\mathbb{R}^4)$, where $\widetilde{Gr}_2(\mathbb{R}^4)$ is the Grassmannian consisting of oriented 2-planes in $\mathbb{R}^4$. Each element of $\widetilde{Gr}_2(\mathbb{R}^4)$ is represented by a 4×2 matrix up to the right action of the group $GL_+(2, \mathbb{R})$ consisting of 2×2 matrices with a positive determinant. We write $[\![a, b]\!]$ for the class represented by a 4×2 matrix (a, b) with rank two.

Putting $y_0 = [0 : 0 : 0 : 1] \in \mathbb{RP}^3$, $\mathbb{RP}^3 - \{y_0\}$ has a line bundle structure over $\mathbb{RP}^2$ defined by $[z_1 : z_2 : z_3 : z_4] \mapsto [z_1 : z_2 : z_3]$, and we denote this line bundle by $\mathcal{O}_{\mathbb{R}}(1)$. Using the Euclidean metric on $\mathbb{R}^3$, $\mathcal{O}_{\mathbb{R}}(1)$ is identified with the tautological bundle $\mathcal{L} = \{([\zeta], v) \in \mathbb{RP}^2 \times \mathbb{R}^3 : v \propto \zeta\}$ by

$$\mathcal{O}_{\mathbb{R}}(1) \ni [t_1 : t_2 : t_3 : \lambda] \longleftrightarrow ([t_1 : t_2 : t_3], \lambda(t_1, t_2, t_3)) \in \mathcal{L}$$

where $t_1^2 + t_2^2 + t_3^2 = 1$.

Let F be the fiber product of the tangent bundle $TS^2 \to S^2$ and $p : \mathbb{P}(TS^2) \to S^2$, and let $\mathcal{O}_{\mathbb{R}}(1) \to \mathbb{RP}^2$ be the tautological bundle. Let $F = L_\parallel \oplus L_\perp$ be the orthogonal decomposition over $\mathbb{P}(TS^2)$ where $L_\parallel = \{(t, w, [v]) \in F : w \propto v\}$ and $L_\perp = \{(t, w, [v]) \in F : w \perp v\}$. We define $\Phi_0 : F \to \mathcal{O}_{\mathbb{R}}(1)$ to be the composition of the orthogonal projection $F \to L_\perp$ with the map $L_\perp \to \mathcal{L} \simeq \mathcal{O}_{\mathbb{R}}(1)$ given by $(t, w, [v]) \mapsto ([t \times v], w)$.

The embedding $TS^2 \to M = \widetilde{Gr}_2(\mathbb{R}^4)$ is given by

$$TS^2 \ni (t, v) \longmapsto \left[\!\!\left[\begin{matrix} t_1 & -v_1 \\ t_2 & -v_2 \\ t_3 & -v_3 \\ 0 & 1 \end{matrix}\right]\!\!\right] \in \widetilde{Gr}_2(\mathbb{R}^4). \tag{6}$$

In this embedding, we have $TS^2 \cong \widetilde{Gr}_2(\mathbb{R}^4) - S^2_\infty$ where S^2_∞ consists of such a point given by

$$\left[\!\!\left[\begin{matrix} t_1 & -v_1 \\ t_2 & -v_2 \\ t_3 & -v_3 \\ 0 & 0 \end{matrix}\right]\!\!\right]. \tag{7}$$

TS^2 is trivial over $D_\pm$ and W, and we define the coordinates $(x_1^\pm, x_2^\pm, x_3^\pm, x_4^\pm) \in \mathbb{R}^4 \cong TS^2|_{D_\pm}$ and $(\alpha, \beta, \varepsilon_1, \varepsilon_2) \in S^1 \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}^2 \cong TS^2|_W$ so as to fit

$$\left[\!\!\left[\begin{matrix} \pm x_1^\pm & -x_4^\pm \\ \pm x_2^\pm & x_3^\pm \\ \pm 1 & 0 \\ 0 & 1 \end{matrix}\right]\!\!\right] \quad \text{and} \quad \left[\!\!\left[\begin{matrix} \cos\alpha & \varepsilon_2 \sin\alpha \\ \sin\alpha & -\varepsilon_2\cos\alpha \\ \tan\beta & \varepsilon_1 \\ 0 & 1 \end{matrix}\right]\!\!\right], \tag{8}$$

in the manner of (6). Then the coordinate change is given by

$$\begin{cases} x_3^\pm = \varepsilon_1 \sin\alpha \cot\beta + \varepsilon_2 \cos\alpha \\ x_4^\pm = -\varepsilon_1 \cos\alpha \cot\beta + \varepsilon_2 \sin\alpha. \end{cases} \tag{9}$$

The coordinates on F are given by $(x_1^\pm, x_2^\pm, x_3^\pm, x_4^\pm, \zeta^\pm)$ and $(\alpha, \beta, \varepsilon_1, \varepsilon_2, \xi)$, and the map Φ_0 is described by

$$\begin{aligned} &\left(x_1^\pm, x_2^\pm, x_3^\pm, x_4^\pm, \zeta^\pm\right) \longmapsto \left[1 : \zeta^\pm : -x_1^\pm - x_2^\pm\zeta^\pm : -x_3^\pm\zeta^\pm + x_4^\pm\right], \\ &(\alpha, \beta, \varepsilon_1, \varepsilon_2, \xi) \longmapsto [\xi\cos\alpha\tan\beta - \sin\alpha : -\xi\sin\alpha\tan\beta + \cos\alpha : \xi : -\varepsilon_1\xi + \varepsilon_2]. \end{aligned} \tag{10}$$

Let $\mathcal{O}(1) \cong \mathbb{CP}^3 - \{y_0\}$ be the complex line bundle defined in a similar way to $\mathcal{O}_{\mathbb{R}}(1)$, and $\mathcal{Z}$ be the fiber product of $TS^2 \to S^2$ and $\mathfrak{p} : \mathbb{D}(TS^2) \to S^2$. Similar to the Zoll case, we obtain the double fibration $TS^2 \leftarrow \mathcal{Z} \to \mathcal{O}(1)$ by extending real parameters $\zeta^\pm$ and ξ to the complex parameters.

The double fibration $M = \widetilde{Gr}_2(\mathbb{R}^4) \leftarrow \hat{\mathcal{Z}} \to \mathbb{CP}^3$ is obtained as follows. For each $x \in M$, we define a holomorphic disk D_x in $\mathbb{CP}^3$ with boundary on $\mathbb{RP}^3$ by the following; $D_x = \Phi_{\mathbb{C},0}(\tilde{\mathfrak{p}}^{-1}(x))$ if $x \in TS^2$, and

$$D_x = \{[z_1 : z_2 : z_3 : u] : \operatorname{Im} u \geq 0\} \cup \{y_0\}$$

if $x \in S^2_\infty$ of the form (7), where $(z_1, z_2, z_3) = t \times v$. Then $\{D_x\}_{x \in M}$ is a family of holomorphic disks in $(\mathbb{CP}^3, \mathbb{RP}^3)$ which foliates $\mathbb{CP}^3 - \mathbb{RP}^3$. Let

$$\hat{\mathcal{Z}} = \{(x, y) \in M \times \mathbb{CP}^3 : y \in D_x\},$$

then $\hat{\mathcal{Z}}$ has a natural smooth structure such that the maps $\wp : \hat{\mathcal{Z}} \to M$ and $\Psi : \hat{\mathcal{Z}} \to \mathbb{CP}^3$ are smooth. In this way, we obtain the double fibration $M \leftarrow \hat{\mathcal{Z}} \to \mathbb{CP}^3$. The real version $M \leftarrow \hat{F} \to \mathbb{RP}^3$ is obtained by restricting $\hat{\mathcal{Z}}$ to the boundary.

Remark 1. By the identification $\widetilde{Gr}_2(\mathbb{R}^4) \cong S^2 \times S^2$, the standard neutral metric g_0 is described in the coordinates of $TD_\pm$ by

$$\frac{1}{1 + \|x\|^2 + \Delta^2} \left(\mathrm{d}x_1^\pm \mathrm{d}x_3^\pm + \mathrm{d}x_2^\pm \mathrm{d}x_4^\pm\right) \quad \text{where} \quad \begin{cases} \|x\|^2 = \sum_{i=1}^4 (x_i^\pm)^2, \\ \Delta = x_1^\pm x_3^\pm + x_2^\pm x_4^\pm. \end{cases}$$

Remark 2. The diagram (5) is regarded as the global version of the diagram given in Figure 1 on page 18 of [4]. On TD_+, the null conformal Killing vector is given by a parallel transport of the coordinates (x_3, x_4).

Remark 3. The point $y = [z_1 : z_2 : z_3 : z_4] \in \mathbb{RP}^3$ corresponds to the embedded two sphere $\{[\![a, b]\!] : ya = yb = 0\} \subset \widetilde{Gr}_2(\mathbb{R}^4)$ which is a β-surface for g_0. On the other hand, the holomorphic disk corresponding to $[\![a, b]\!] \in \widetilde{Gr}_2(\mathbb{R}^4)$ is the closure of one of the two connected components of $\{y \in \mathbb{CP}^3 - \mathbb{RP}^3 : ya = yb = 0\}$.

3. Definition of the singularity

In this section, we introduce a certain singularity of metrics and twistor spaces, and state a conjecture for a singular version of the twistor correspondence. The main purpose in this article is to construct explicit examples such that the conjecture holds, restricted to these examples. That is explained in the following sections.

First, we study the twistor spaces for the self-dual Zolfrei metrics. In the non-singular case, as explained in Section 2, the twistor space is given by a pair $(\mathbb{CP}^3, P)$ where P is the image of a totally real embedding $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$. In [11], LeBrun and Mason proved that if the embedding ι is close to the standard one, then there is a unique smooth family of holomorphic disks in $\mathbb{CP}^3$ with boundaries on P which satisfy that (1) the relative homology class of each disk generates $H_2(\mathbb{CP}^3, P; \mathbb{Z}) \cong \mathbb{Z}$, and (2) interiors of these disks smoothly foliate $\mathbb{CP}^3 - P$.

We formulate the singular version of the above conditions. Let $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$ be the continuous injection such that the restriction $\mathbb{RP}^3 - \{y_0\} \to \mathbb{CP}^3$ is a totally real embedding, and let $P = \iota(\mathbb{RP}^3)$. We assume ι is homotopic to the standard embedding $\mathbb{RP}^3 \subset \mathbb{CP}^3$, then we have $H_2(\mathbb{CP}^3, P; \mathbb{Z}) \cong \mathbb{Z}$.

Definition 4. The pair $(\mathbb{CP}^3, P)$ is said to satisfy the condition (♯) if there is a unique family of holomorphic disks in $\mathbb{CP}^3$ with boundaries on P which satisfy that

(♯ 1) the relative homology class of each disk generates $H_2(\mathbb{CP}^3, P; \mathbb{Z})$,
(♯ 2) interiors of these disks foliate $\mathbb{CP}^3 - P$, and the disks which do not contain y_0 forms a smooth family.

In Sections 5 and 6, we construct examples of the pair $(\mathbb{CP}^3, P)$ which is given by such singular embeddings and which satisfy the condition (♯).

Next, we define the 'singular self-dual Zollfrei metric'. Let M be a four-dimensional smooth manifold, C be a two-dimensional closed submanifold of M, and g be a neutral metric on $M - C$.

Definition 5. g is called a singular neutral metric with a singular β-surface C if, for all $x \in C$, there is an open neighborhood $x \in U \subset M$ with the coordinate $u = (u_1, u_2, u_3, u_4)$ on U, which satisfies the following two conditions:

(1) $C \cap U = \{u \in U : u_2 = u_3 = 0\}$.

(2) Let $j : (u_1, u_4, r, \phi) \mapsto (u_1, u_2, u_3, u_4)$ be the cylindrical coordinate given by $u_2 = r\cos\phi$, $u_3 = r\sin\phi$. Then, for some smooth function h on $U \setminus C$, we can write $hg = g_{\mathrm{st}} + g_1 + o(r)$ such that $g_{\mathrm{st}} = 2(\mathrm{d}u_1\mathrm{d}u_3 + \mathrm{d}u_2\mathrm{d}u_4)$ is the standard neutral metric, g_1 is a symmetric tensor satisfying $j^*g_1 = \rho(u_1, u_4, \phi)r^2\mathrm{d}\phi^2$ for some function ρ, and $o(r)$ is the error term satisfying $\lim_{r\to 0} o(r) = 0$.

We can also define the *singular neutral conformal structure* on M with singular β-surface C.

Remark 6. If we take the limit $r \to 0$ for fixed (u_1, u_4, ϕ), then $g_0 + g_1 + o(r) \to g_0 + \rho\cdot(-\sin\phi\mathrm{d}u_2 + \cos\phi\mathrm{d}u_3)^2$. This limit defines a neutral metric on T_xM where $x = \lim u \in C$. This metric depends on ϕ, but $\{u_2 = u_3 = 0\}$ always defines a β-plane. This is the reason why we call C a 'singular β-surface'.

Definition 7. (i) A neutral metric g on $M - C$ is called a *singular self-dual neutral metric* on M when

(1) g is self-dual on $M - C$, and
(2) g is a singular neutral metric with singular β-surface C.

(ii) g is called a *singular Zollfrei metric* when every null geodesic in $M - C$ satisfies either of the following; it is closed, or the ends of its closure in M are the points in C which are not necessarily distinct.

In the non-singular case, every β-surface is either S^2 or $\mathbb{RP}^2$ if the metric g is self-dual Zollfrei (Theorem 5.14 of [11]). In our case, the β-surface is defined only on $M - C$. However, we will see later that, in our examples, the closure of each β-surface in M is homeomorphic to S^2, with two extra points (Propositions 16 and 23). Motivated by these examples, we state the following conjectures.

Conjecture 8. *Let g be a singular self-dual Zollfrei metric on M with a singular β-surface C. Let S be any β-surface in $M - C$, and $\overline{S}$ be its closure in M. Then $\overline{S} - S$ is a finite subset of C and $\overline{S}$ is a topological manifold.*

Let g and S be as in the above conjecture. We simply call $\overline{S}$ a β-surface for g.

Conjecture 9. *There is a natural one-to-one correspondence between*

- *equivalence classes of singular self-dual Zollfrei conformal structures on $S^2 \times S^2$ with one singular β-surface $C \simeq S^2$,*
- *equivalence classes of the pairs $(\mathbb{CP}^3, P)$ which satisfy the condition $(\sharp)$, where P is the image of a totally real embedding $\mathbb{RP}^3 \to \mathbb{CP}^3$ which has one singular point.*

This correspondence should be characterized by the similar condition as in LeBrun and Mason's theorems, and the explicit formulation for our examples is given in Theorem 24. We formulated the conjecture for the simplest possible singularity of the twistor space because otherwise, more complicated phenomenon would occur such as the intersection of two singular β-surfaces.

4. Construction of singular metrics

In this section, we construct the examples of the singular self-dual Zollfrei metric, by patching a pair of Petean's indefinite self-dual metrics. We will see later that each example corresponds to an element of the set $\mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ consisting of functions that are rapidly decreasing, axisymmetric, and real valued. Recall that a smooth function $f(x)$ on $\mathbb{R}^n$ is called *rapidly decreasing* if and only if for each polynomial P and each integer $m \geq 0$

$$\sup \left||x|^m P(\partial_1, \ldots, \partial_n) f(x)\right| < \infty$$

where $|x|$ denotes the norm of x. We write $\mathcal{S}(\mathbb{R}^n)$ for the set consisting of rapidly decreasing real valued functions on $\mathbb{R}^n$. We call $f \in \mathcal{S}(\mathbb{R}^2)$ *axisymmetric* if and only if f is $SO(2)$-invariant.

Petean's metric is an indefinite metric over $\mathbb{R}^4$ of the form

$$g = 2(\mathrm{d}x_1\mathrm{d}x_3 + \mathrm{d}x_2\mathrm{d}x_4) + f(x_1, x_2)\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right), \tag{11}$$

where f is a smooth function. Such metrics are first introduced by J. Petean to construct examples of the indefinite Kähler–Einstein metric on the complex tori or the primary Kodaira surfaces [14]. For the metric (11), we have an indefinite orthonormal frame $\{e_i\}_i$ given by

$$(e_1, e_2, e_3, e_4) = (\partial_1, \partial_2, \partial_3, \partial_4)\, \frac{1}{2\sqrt{D}} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -b & b & a & -a \\ -b & -b & a & a \end{pmatrix},$$

where $\partial_i = \frac{\partial}{\partial x_i}$ and

$$D = f^2 + 4, \qquad a = \frac{f + \sqrt{D}}{2}, \qquad b = \frac{f - \sqrt{D}}{2}.$$

Let $\wedge^2_- T\mathbb{R}^4$ be the bundle of anti-self-dual bivectors. If we trivialize $\wedge^2_- T\mathbb{R}^4$ by the frame

$$e_1 \wedge e_2 - e_3 \wedge e_4, \qquad e_1 \wedge e_3 - e_2 \wedge e_4, \qquad e_1 \wedge e_4 + e_2 \wedge e_3,$$

then we can check that the induced Levi-Civita connection is trivial.

Remark 10. Petean's metric is a special case of the *Walker metric* (cf. [12,13]). Furthermore, the orthonormal frame defined above looks similar to that explained in [12].

Following the arguments in [11], the vectors

$$\begin{aligned} \mathfrak{m}_1 &= e_1 - \sin 2\sigma\ e_3 + \cos 2\sigma\ e_4 \\ \mathfrak{m}_2 &= e_1 + \cos 2\sigma\ e_3 + \sin 2\sigma\ e_4 \end{aligned} \tag{12}$$

span the β-plane at every point in $\mathbb{R}^4$ for each $\sigma \in \mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$. If we put

$$\begin{aligned} \mathfrak{n}_1 &= \cos\sigma\ \partial_1 + \sin\sigma\ \partial_2 - \frac{f}{2}\left(\cos\sigma\ \partial_3 + \sin\sigma\ \partial_4\right), \\ \mathfrak{n}_2 &= \sin\sigma\ \partial_3 - \cos\sigma\ \partial_4, \end{aligned} \tag{13}$$

then the distribution $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle$ is equal to $\langle \mathfrak{n}_1, \mathfrak{n}_2 \rangle$.

Proposition 11. *Let g be a Petean's metric of the form (11), then every β-surface on $(\mathbb{R}^4, g)$ is given by the solutions of*

$$\begin{cases} -\sin\sigma\ x_1 + \cos\sigma\ x_2 = c_1, \\ \cos\sigma\ x_3 + \sin\sigma\ x_4 + \varphi(x_1, x_2, \sigma) = c_2, \end{cases} \tag{14}$$

for some real constants σ, c_1 and c_2, where

$$\varphi(x_1, x_2, \sigma) = \frac{1}{2}\int_0^{\lambda} f(\cos\sigma\ t - \sin\sigma\ \mu, \sin\sigma\ t + \cos\sigma\ \mu)\mathrm{d}t, \tag{15}$$

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Proof. Notice that φ satisfies

$$\cos\sigma\ \frac{\partial\varphi}{\partial x_1} + \sin\sigma\ \frac{\partial\varphi}{\partial x_2} = \frac{f}{2}. \tag{16}$$

Therefore, the left-hand sides of (14) are both annihilated by $\mathfrak{n}_1$ and $\mathfrak{n}_2$. Hence these are constant along some β-surface. □

Because (σ, c_1, c_2) and $(\sigma + \pi, -c_1, -c_2)$ correspond to the same β-surface, we can assume $\sigma \in [0, \pi)$.

Remark 12. In [2], Blair et al. showed that the 'hyperbolic twistor space' over $\mathbb{R}^4$ equipped with a Petean's metric is holomorphically trivial. They proved this fact by constructing an explicit complex coordinate of the twistor space. This construction actually works in the case of the 'reflector space' [8] or, in the other literature, the 'product twistor space' [1]. In this way, we obtain essentially the same statement as Proposition 11.

Remark 13. One can show that a Petean's metric in the form (11) is flat if and only if f is harmonic (i.e. $f_{x_1x_1} + f_{x_2x_2} = 0$). Hence, for example, if we assume f is rapidly decreasing, then the metric is flat if and only if $f \equiv 0$ (cf. [13,14]).

Definition 14. Let g be a Petean's metric of the form (11), then g is called *rapidly decreasing* (respectively, compact supported, or axisymmetric) if and only if f is so. On the other hand, the *dual of* g is another Petean's metric of the form

$$g^\vee = 2(\mathrm{d}x_1\mathrm{d}x_3 + \mathrm{d}x_2\mathrm{d}x_4) - f(x_1, x_2)\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right).$$

Remark 15. In the space of Petean's metrics of the form (11), the flat metric, i.e. the case of $f \equiv 0$, is characterized by an $(S^1 \times S^1)$-invariance, where $(\tau_1, \tau_2) \in S^1 \times S^1$ acts on $\mathbb{R}^4$ by

$$\begin{pmatrix} x_1 & -x_4 \\ x_2 & x_3 \end{pmatrix} \longmapsto R(\tau_1)\begin{pmatrix} x_1 & -x_4 \\ x_2 & x_3 \end{pmatrix} R(\tau_2)^{-1}, \quad \left(R(\tau) = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}\right).$$

In the same way, the axisymmetric metrics are characterized by the $(S^1 \times \{1\})$-invariance. On the other hand, the 'dual' defines a $\mathbb{Z}_2$-action on the space of Petean's metrics, then the flat metric is also characterized by the $\mathbb{Z}_2$-invariance.

We use the coordinates introduced in Section 2 from now on.

Proposition 16. *Let $g_\pm$ be compact supported Petean's metrics on $TD_\pm$ given by*

$$g_\pm = 2(\mathrm{d}x_1^\pm \mathrm{d}x_3^\pm + \mathrm{d}x_2^\pm \mathrm{d}x_4^\pm) + f_\pm(x_1^\pm, x_2^\pm)\left((\mathrm{d}x_1^\pm)^2 + (\mathrm{d}x_2^\pm)^2\right). \tag{17}$$

Then the conformal structures of these metrics extend to a self-dual indefinite conformal structure on TS^2. Moreover, if $g_\pm$ are both axisymmetric and dual each other, then every β-surface on TS^2 is an embedded $S^1 \times \mathbb{R}$ whose closure in $\widetilde{Gr}_2(\mathbb{R}^4)$ is homeomorphic to S^2, with two extra points.

Proof. If $g_\pm$ are flat, i.e. $f_\pm \equiv 0$, it is obvious that these conformal structures extend to the standard conformal structure on $\widetilde{Gr}_2(\mathbb{R}^4)$ by the remark in Section 2. In the general case, because we assumed $f_\pm$ are compact supported, these metrics are flat in some neighborhood of $TS|_{W_0}$ where $W_0 = \{(\alpha, \beta) \in W : \beta = 0\}$. Therefore, these conformal structures extend to a self-dual conformal structure on TS^2.

Now we suppose the axisymmetricity and the duality, and denote simply $f_+ = -f_- = f$. Because f is compact supported, we can take $R > 0$ such that $f(x_1^+, x_2^+) = 0$ when $(x_1^+)^2 + (x_2^+)^2 > R^2$. By the Proposition 11, each β-surface on $TD_\pm$ is given by

$$\begin{cases} -\sin\sigma^\pm\, x_1^\pm + \cos\sigma^\pm\, x_2^\pm = c_1^\pm, \\ \cos\sigma^\pm\, x_3^\pm + \sin\sigma^\pm\, x_4^\pm \pm \varphi(x_1^\pm, x_2^\pm, \sigma^\pm) = c_2^\pm, \end{cases} \tag{18}$$

for some $(\sigma^\pm, c_1^\pm, c_2^\pm)$, where φ is defined by (15). We observe that when $(\sigma^+, c_1^+, c_2^+) = (\sigma^-, c_1^-, c_2^-)$, corresponding β-surfaces are nicely extended in TS^2. Therefore, we drop the signs on σ, c_1, c_2 for simplicity. Changing the coordinates by (3) and (9), the β-surfaces are described in the regions $\{(\alpha, \beta, \varepsilon_1, \varepsilon_2) \in TW : 0 < |\tan\beta| < R^{-1}\}$ by

$$\begin{cases} \sin(\alpha - \sigma) = c_1 \tan\beta, \\ -c_1\varepsilon_1 - \cos(\alpha - \sigma)\varepsilon_2 + \psi(\alpha, \beta, \sigma) = c_2, \end{cases} \tag{19}$$

where ψ is defined as the following. Let

$$\hat{f}^{(\pm)}(\sigma, \mu) = \int_0^{\pm\infty} f(\cos\sigma\, t - \sin\sigma\, \mu,\ \sin\sigma\, t + \cos\sigma\, \mu)\mathrm{d}t$$

be the 'half Radon transform' of f. Then we define ψ, when $0 < \tan\beta < R^{-1}$,

$$\psi(\alpha,\beta,\sigma) = \begin{cases} \frac{1}{2}\hat{f}^{(+)}(\sigma, \sin(\alpha-\sigma)\cot\beta) & \text{if } \cos(\alpha-\sigma) > 0 \\ \frac{1}{2}\hat{f}^{(-)}(\sigma, \sin(\alpha-\sigma)\cot\beta) & \text{if } \cos(\alpha-\sigma) < 0 \end{cases}$$

and when $-R^{-1} < \tan\beta < 0$,

$$\psi(\alpha,\beta,\sigma) = \begin{cases} -\frac{1}{2}\hat{f}^{(-)}(\sigma, \sin(\alpha-\sigma)\cot\beta) & \text{if } \cos(\alpha-\sigma) > 0 \\ -\frac{1}{2}\hat{f}^{(+)}(\sigma, \sin(\alpha-\sigma)\cot\beta) & \text{if } \cos(\alpha-\sigma) < 0. \end{cases}$$

Let $\hat{f}$ be the Radon transform explained in the Appendix, then we have $\hat{f}^{(+)} = -\hat{f}^{(-)} = \frac{1}{2}\hat{f}$ from the axisymmetricity. Moreover, $\hat{f}(\mu) = \hat{f}(\sigma,\mu)$ is even for μ and does not depend on σ. Therefore, Eq. (19) extends to $|\tan\beta| < R^{-1}$ by exchanging $\psi(\alpha,\beta,\sigma)$ with

$$\tilde{\psi}(\alpha,\sigma;c_1) = \begin{cases} \frac{1}{4}\hat{f}(c_1) & \text{if } \cos(\alpha-\sigma) > 0 \\ -\frac{1}{4}\hat{f}(c_1) & \text{if } \cos(\alpha-\sigma) < 0. \end{cases} \tag{20}$$

Notice that $\cos(\alpha-\sigma) \neq 0$ near $\{\beta = 0\}$ because $\sin(\alpha-\sigma)$ is close to zero by the first equation of (19). Hence we have proved that a suitable pair of β-surfaces on $TD_\pm$ extends to a β-surface on TS^2. This β-surface has a fiber bundle structure with respect to the projection $TS^2 \to S^2$. In fact the base space is a big circle on S^2 and the fibers are orientable linear lines in some tangent space of S^2. Therefore, it is isomorphic to a cylinder $S^1 \times \mathbb{R}$. If we take any point on this β-surface, and let it move away along the fiber, then the point goes to

$$\left[\!\!\left[\begin{matrix} \mp c_1\sin\sigma & \cos\sigma \\ \pm c_1\cos\sigma & \sin\sigma \\ \pm 1 & 0 \\ 0 & 0 \end{matrix}\right]\!\!\right] \in S^2_\infty, \tag{21}$$

so the proof is completed. □

The two points (21) are actually *antipodal*, i.e. they are exchanged by the natural involution of S^2_∞ which is defined as a deck transformation for the orientation forgetting map $\widetilde{Gr}_2(\mathbb{R}^4) \to Gr_2(\mathbb{R}^4)$.

Proposition 17. *Let $g_\pm$ be Petean's metrics on $TD_\pm$ in the form (17), and we assume that $g_\pm$ are compact supported, axisymmetric and dual each other. Then the induced self-dual conformal structure on $TS^2 = \widetilde{Gr}_2(\mathbb{R}^4) - S^2_\infty$ defines a singular self-dual conformal structure on $\widetilde{Gr}_2(\mathbb{R}^4)$ in the sense of Definition 5.*

Proof. We check that S^2_∞ is a singular β-surface. As an example, we study the coordinate neighborhood U given by

$$\mathbb{R}^4 \ni (u_1,u_2,u_3,u_4) \longmapsto \left[\!\!\left[\begin{matrix} u_1 & -u_4 \\ 1 & 0 \\ 0 & 1 \\ u_2 & u_3 \end{matrix}\right]\!\!\right] \in \widetilde{Gr}_2(\mathbb{R}^4).$$

In this coordinate, S^2_∞ is given by $\{u_2 = u_3 = 0\}$. We can check that $U \cap TD_+$ is given by $\{x \in TD_+ : x_3^+ < 0\} = \{u \in U : u_2 < 0\}$, and by a direct calculation,

$$g_+ = \frac{1}{u_2^2}\left[2(\mathrm{d}u_1\mathrm{d}u_3 + \mathrm{d}u_2\mathrm{d}u_4) + r^2 f(u_1\tan\phi - u_4, -\tan\phi)\left\{(1+u_1^2)\mathrm{d}\phi^2 + (\sin\phi\,\mathrm{d}u_1 + \cos\phi\,\mathrm{d}u_4)^2\right\}\right]$$

where $u_2 = r\cos\phi$, $u_3 = r\sin\phi$. If we put $\rho = f\cdot(1+u_1^2)$ and so on, the condition (2) of Definition 5 is satisfied.

In this way, checking several coordinate neighborhoods, we can show that S^2_∞ is a singular β-surface for the metric. □

Remark 18. The $(S^1 \times S^1)$-action on $TD_\pm$ extends to $\widetilde{Gr}_2(\mathbb{R}^4)$ by

$$[\![a, b]\!] \longmapsto [\![\tilde{R}(\tau_1, \tau_2)(a, b)]\!] \quad \left(\tilde{R}(\tau_1, \tau_2) = \begin{pmatrix} R(\tau_1) & O \\ O & R(\tau_2) \end{pmatrix}\right).$$

The standard metric on $\widetilde{Gr}_2(\mathbb{R}^4)$ is invariant under this action, and our singular metric is invariant under $(S^1 \times \{1\})$-action.

Next, we check the Zollfrei condition for our singular metric. In the non-singular case, self-dual metric g is Zollfrei if every β-surface is an embedded S^2 (Theorem 5.14. of [11]). Although we cannot apply this theorem in our case, we can check the condition directly by writing down null geodesics explicitly.

First, the following formulae are checked by a direct calculation:

Lemma 19. *Let ∇ be the Levi-Civita connection of a Petean's metric of the form (11) and let $D = f^2 + 4$, then*

$$\begin{aligned} &\nabla_{\partial_3} = \nabla_{\partial_4} = 0, \\ &\nabla_{\partial_1}\partial_3 = \frac{f\partial_1 f}{2D}\partial_3, \qquad \nabla_{\partial_1}\partial_4 = \frac{f\partial_1 f}{2D}\partial_4, \\ &\nabla_{\partial_1}\partial_1 = \frac{f\partial_1 f}{2D}\partial_1 + \frac{\partial_1 f}{2}\partial_3 - \frac{\partial_2 f}{2}\partial_4. \end{aligned} \tag{22}$$

Proposition 20. *The singular self-dual metric on $\widetilde{Gr}_2(\mathbb{R}^4)$ in Proposition 17 is singular Zollfrei.*

Proof. Because every null geodesic is contained in some β-surface, the image of a null geodesic by the projection $TS^2 \to S^2$ is contained in some big circle of S^2. We prove that this image is either one point or a whole circle. If the image is one point, from (18) or (19), the null geodesic must be a linear line in some tangent space of S^2. This is actually a geodesic, because, for example on TD_+, we have $\nabla_{\partial_3} = \nabla_{\partial_4} = 0$ by Lemma 19. Notice that the end points of such a linear line are the antipodal points on S^2_∞.

For the other case, we first study about a null geodesic in TD_+ whose image on D_+ is not one point. Without any loss of generality, we can assume this geodesic is contained in a β-surface with $\sigma^+ = 0$ in (18) because of the axisymmetricity. Let $c(s)$ be a curve contained in this β-surface. If the projected image on D_+ is not one point, such a curve is always written in the form

$$x_1^+ = s, \qquad x_2^+ = c_1, \qquad x_3^+ = c_2 - \varphi(s, c_1, 0), \qquad x_4^+ = \nu(s),$$

at least for a small interval of parameter s, where $\nu(s)$ is an unknown function. Then the velocity vector is

$$\dot{c}(s) = \partial_1 - \frac{f}{2}\partial_3 + \frac{\partial \nu}{\partial s}\partial_4.$$

Using Lemma 19, we have

$$\tilde{\nabla}_{\frac{\partial}{\partial s}}\dot{c}(s) = \frac{f\partial_1 f}{2D}\dot{c}(s) + \left(\frac{\mathrm{d}^2\nu}{\mathrm{d}s^2} - \frac{\partial_2 f}{2}\right)\partial_4,$$

where $\tilde{\nabla}$ is the covariant derivative along $c(s)$. Hence $c(s)$ is an unparameterized geodesic if and only if

$$\frac{\mathrm{d}^2\nu}{\mathrm{d}s^2} = \frac{1}{2}\partial_2 f(s, c_1). \tag{23}$$

Let $\nu_0(s)$ be the solution of (23) with $\nu_0(0) = \nu_0'(0) = 0$, then any solution of (23) is given by

$$\nu(s) = \nu_0(s) + q_1 s + q_2,$$

for some constants q_1 and q_2. Every null geodesic on TD_+ whose image to D_+ is not one point is given by rotating the above $c(s)$. Note that ν_0 is an even function, moreover, because f is compact supported, ν_0 is a degree one polynomial for $|s| \gg 0$. Therefore, we have

$$\nu_0(s) = A_1|s| + A_2 \quad (|s| > R), \tag{24}$$

where A_1 and A_2 are constants, and R is a large constant such as $f(x) \equiv 0$ for $|x| > R$.

Now notice the following geodesics on $TD_\pm$ parameterized by $s_\pm$, respectively:

$$x_1^\pm = s_\pm, \qquad x_2^\pm = c_1, \qquad x_3^\pm = c_2 \mp \varphi(s_\pm, c_1, 0), \qquad x_4^\pm = \pm\nu_0(s_\pm) + q_1^\pm s_\pm + q_2^\pm.$$

We prove that these geodesics extend smoothly in TS^2 for suitable $q_1^\pm$ and $q_2^\pm$, by changing the parameter

$$u_+ = \begin{cases} -\dfrac{1}{s_+} & s_+ > 0 \\ -\dfrac{1}{s_-} & s_- < 0, \end{cases} \qquad u_- = \begin{cases} -\dfrac{1}{s_-} & s_- > 0 \\ -\dfrac{1}{s_+} & s_+ < 0. \end{cases}$$

Changing the coordinates by (3) and (9), these geodesics are written on $u_+ < 0$

$$\alpha = \tan^{-1}(-c_1 u_+) \quad \left(-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}\right), \qquad \beta = \tan^{-1}\left(-u_+(1 + c_1^2 u_+^2)^{-\frac{1}{2}}\right),$$

$$\varepsilon_1 = \frac{1}{1 + c_1^2 u_+^2}\left(c_1 u_+^2 \Theta + \Xi\right), \qquad \varepsilon_2 = \frac{1}{(1 + c_1^2 u_+^2)^{\frac{1}{2}}}\left(\Theta - c_1 \Xi\right),$$

where

$$\Theta = c_2 - \varphi\left(-u_+^{-1}, c_1, 0\right), \qquad \Xi = u_+ \nu_0\left(-u_+^{-1}\right) - q_1^+ + q_2^+ u_+.$$

Similarly on $u_+ > 0$, $(\alpha, \beta, \varepsilon_1, \varepsilon_2)$ are given by the same equation by exchanging

$$\Theta = c_2 + \varphi\left(-u_+^{-1}, c_1, 0\right), \qquad \Xi = -u_+ \nu_0\left(-u_+^{-1}\right) - q_1^- + q_2^- u_+.$$

By similar argument in the proof of Proposition 16, Θ extends smoothly to $u_+ = 0$, and from (24), Ξ extends smoothly to $u_+ = 0$ iff $q_1^- = q_1^+$ and $q_2^- = q_1^+ + 2A_1$. By similar argument for u_-, we obtain the same conditions. Therefore, these geodesics are nicely extended to a closed curve if $q_1^- = q_1^+$ and $q_2^- = q_1^+ + 2A_1$.

The rest possibility is that the projected image of a null geodesic is contained in the equator W_0. However, these null geodesics are, of course, closed, because the conformal structure is standard around $TS^2|_{W_0}$. □

So far, we assumed that the Petean's metrics are all compact supported, but this assumption is weakened to be 'rapidly decreasing'. Actually the argument of the extension to $TS^2|_{W_0}$ works well essentially in the same manner, and only one thing that we have to check is the smoothness at infinity, which is almost obvious from the rapidly decreasing condition. In all, we have

Theorem 21. *Let g_+ and g_- be rapidly decreasing Petean's metrics on $\mathbb{R}^4$, then the disjoint union $(\mathbb{R}^4, [g_+]) \amalg (\mathbb{R}^4, [g_-])$ naturally extends to a self-dual indefinite conformal structure on $\widetilde{Gr}_2(\mathbb{R}^4) - S^2_\infty$. Moreover, if $g_\pm$ are both axisymmetric and dual with each other, then this conformal structure defines a singular self-dual Zollfrei conformal structure on $\widetilde{Gr}_2(\mathbb{R}^4)$ with singular β-surface S^2_∞.*

Remark 22. These are the required examples. Notice that each singular self-dual Zollfrei conformal structure corresponds to a function $f \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ by using the Petean's metric g_+ which corresponds to f.

From Proposition 16, we also have

Proposition 23. *Conjecture 8 holds for the singular self-dual Zollfrei conformal structure corresponding to $f \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$.*

5. Construction of singular twistor spaces

Recall that we identify $\mathbb{RP}^3 - \{y_0\}$ with the line bundle $\mathcal{O}_\mathbb{R}(1) \to \mathbb{RP}^2$, where $y_0 = [0:0:0:1]$. Let $\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$ be the subset of $\mathcal{S}(\mathbb{R})$ consisting of odd functions. For each $s(t) \in \mathcal{S}(\mathbb{R})^{\mathrm{odd}}$, we define a smooth section $\tilde{s}$ of the line bundle $\mathcal{O}_\mathbb{R}(1) \to \mathbb{RP}^2$ by $\tilde{s}([0:0:1]) = [0:0:1:0]$ and

$$\tilde{s}\left(\left[-\sin\frac{\theta}{2} : \cos\frac{\theta}{2} : t\right]\right) = \left[-\sin\frac{\theta}{2} : \cos\frac{\theta}{2} : t : s(t)\right]. \tag{25}$$

Then we put

$$P=\mathcal{O}_{\mathbb{R}}(1)+\mathrm{i}\,\tilde{s}(\mathbb{RP}^2)=\left\{u+\mathrm{i}\,\tilde{s}(x)\in\mathcal{O}(1):x\in\mathbb{RP}^2,u\in\mathcal{O}_{\mathbb{R}}(1)_x\right\},$$

which is a deformation of $\mathcal{O}_{\mathbb{R}}(1)$ in $\mathcal{O}(1)$. Adding the point y_0 to the pair $(\mathcal{O}(1),P)$, we have the twistor space $(\mathbb{CP}^3,\hat{P})$. Notice that $\hat{P}$ is the image of a map $\iota:\mathbb{RP}^3\to\mathbb{CP}^3$ given by $\iota(y_0)=y_0$ and

$$\mathcal{O}_{\mathbb{R}}(1)\ni u\longmapsto u+\mathrm{i}\,\tilde{s}(\pi(u))\in P$$

using the projection π. For later convenience, we use the pure imaginary valued function $h(t)=\mathrm{i}s(t)\in\mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$ from now on. Then our goal is:

Theorem 24. *There is a natural one-to-one correspondence between*

- *singular self-dual Zollfrei conformal structures on $\widetilde{Gr}_2(\mathbb{R}^4)$ corresponding to $f(x)\in\mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$,*
- *the pairs $(\mathbb{CP}^3,\hat{P})$ corresponding to $h(t)\in\mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$,*

which satisfies the following property. There is a double fibration $\widetilde{Gr}_2(\mathbb{R}^4)\xleftarrow{\wp}\hat{\mathcal{Z}}\xrightarrow{\Psi}\mathbb{CP}^3$ such that

(1) *Ψ is a continuous surjection and $\wp$ is a complex disk bundle, i.e. for each $x\in\widetilde{Gr}_2(\mathbb{R}^4)$, $\hat{\mathcal{Z}}_x=\wp^{-1}(x)$ is biholomorphic to the complex unit disk,*
(2) *$\Psi_x:\hat{\mathcal{Z}}_x\to\mathbb{CP}^3$ is holomorphic on the interior of $\hat{\mathcal{Z}}_x$ and $\Psi_x(\partial\hat{\mathcal{Z}}_x)\subset\hat{P}$, where Ψ_x is the restriction of Ψ on $\hat{\mathcal{Z}}_x$,*
(3) *the restriction of Ψ on $\hat{\mathcal{Z}}-\partial\hat{\mathcal{Z}}$ is bijective onto $\mathbb{CP}^3-\hat{P}$,*
(4) *$\{\wp(\Psi^{-1}(y))\}_{y\in P}$ is equal to the set of β-surfaces on $\widetilde{Gr}_2(\mathbb{R}^4)$.*

Moreover, this correspondence is explicitly given by the formulae $f=2\mathrm{i}\left(\frac{\partial}{\partial t}h\right)^{\vee}$ and $h=-\frac{1}{4}\mathcal{H}\hat{f}$.

Remark 25. The dual Radon transform $(\cdot)^{\vee}$ and the Hilbert transform $\mathcal{H}$ are explained in the Appendix.

Here we explain some reasons why the above construction of the twistor space is reasonable. Firstly, recall that our singular metrics are standard on the equator W_0, Therefore, the twistor spaces would be standard over the corresponding point $[0:0:1]\in\mathbb{RP}^2$. This corresponds to $\tilde{s}([0:0:1])=[0:0:1:0]$. Moreover, our singular metrics have S^1-symmetry, Therefore, the twistor spaces would also have a similar symmetry. From the twistor correspondence for the standard Zoll projective structure, the S^1-action on $\mathbb{RP}^2$ is given by

$$S^1\ni\tau:[z_1:z_2:z_3]\mapsto[\cos\tau\,z_1-\sin\tau\,z_2:\sin\tau\,z_1+\cos\tau\,z_2:z_3].$$

And the lift of this S^1-action on $\mathcal{O}_{\mathbb{R}}(1)=\mathbb{RP}^3-\{y_0\}$ is given by

$$[z_1:z_2:z_3:z_4]\mapsto[\cos\tau\,z_1-\sin\tau\,z_2:\sin\tau\,z_1+\cos\tau\,z_2:z_3:z_4].$$

Therefore, $\tilde{s}$ should be S^1-equivariant, if the twistor space is given by section $\tilde{s}$ of $\mathcal{O}_{\mathbb{R}}(1)$, and if it corresponds to our singular metric. Such a section is given by an odd function $s(t)$ in the manner of (25).

Before we start to prove Theorem 24, we remark, firstly, that we study the real twistor correspondence for our singular self-dual Zollfrei metric. This real correspondence is important not only for its geometric significance but also as a step in the construction of the complex correspondence. We define a map $\Phi:F\to\mathcal{O}_{\mathbb{R}}(1)$ by

$$\begin{aligned}
&F|_{D_\pm}\ni\left(x_1^\pm,x_2^\pm,x_3^\pm,x_4^\pm,\zeta\right)\longmapsto\left[1:\zeta:-x_1^\pm-x_2^\pm\zeta:-x_3^\pm\zeta+x_4^\pm\pm\frac{\varphi}{\sin\sigma}\right],\\
&F|_{W_0}\ni(\alpha,0,\varepsilon_1,\varepsilon_2,\xi)\longmapsto\left[-\sin\alpha:\cos\alpha:\xi:-\varepsilon_1\xi+\varepsilon_2+\frac{1}{4}\hat{f}(\xi)\right],
\end{aligned}\tag{26}$$

where $\zeta=-\cot\sigma$ and $W_0=\{\beta=0\}\subset W$. From (18) and (19), the inverse image of a point by this map is a β-surface. Hence $\mathcal{O}_{\mathbb{R}}(1)$ is identified with the space of β-surfaces by this map.

Next, we study the complex structure on $\mathcal{Z}$, where $\mathcal{Z}$ is given in diagram (5). $\mathcal{Z}$ has a natural complex structure defined from the self-dual metric on TS^2 as the following (cf. [11], see also [2]). Recall that for a given Petean's metric, any β-plane at a point is written in the form $\langle \mathfrak{n}_1, \mathfrak{n}_2 \rangle$ where $\mathfrak{n}_1, \mathfrak{n}_2$ is given by (13). Putting $\zeta = -\cot\sigma$ and by the 'analytic continuation', we obtain the complex tangent vectors

$$\mathfrak{n}_1' = -\zeta^+ \frac{\partial}{\partial x_1^+} + \frac{\partial}{\partial x_2^+} - \frac{f}{2}\left(-\zeta^+ \frac{\partial}{\partial x_3^+} + \frac{\partial}{\partial x_4^+}\right), \qquad \mathfrak{n}_2' = \frac{\partial}{\partial x_3^+} + \zeta^+ \frac{\partial}{\partial x_4^+},$$

on $\mathcal{Z}|_{TD_+}$. The complex structure on $\mathcal{Z}|_{TD_+}$ is defined so that its (0,1)-vector space is spanned by $\mathfrak{n}_1', \mathfrak{n}_2'$ and $\frac{\partial}{\partial \zeta^+}$.

The key to prove Theorem 24 is to construct a map $\Phi_{\mathbb{C}} : \mathcal{Z} \to \mathcal{O}(1)$ which is described in the form

$$\left(x_1^+, x_2^+, x_3^+, x_4^+, \zeta^+\right) \longmapsto \left[1 : \zeta^+ : -x_1^+ - x_2^+\zeta^+ : -x_3^+\zeta^+ + x_4^+ + H(x_1^+, x_2^+, \zeta^+)\right]$$

on D_+. We will construct the surjection $\Psi : \hat{\mathcal{Z}} \to \mathbb{CP}^3$ as an extension of $\Phi_{\mathbb{C}}$. Notice that $\Phi_{\mathbb{C}}$ is holomorphic on $\mathcal{Z}|_{D_+}$ with respect to the above complex structure if and only if (i) $H(x_1^+, x_2^+, \zeta^+)$ is holomorphic for ζ^+ and (ii) H solves

$$\left(-\zeta^+ \frac{\partial}{\partial x_1^+} + \frac{\partial}{\partial x_2^+}\right) H(x_1^+, x_2^+, \zeta^+) = \frac{f(x_1^+, x_2^+)}{2}\left((\zeta^+)^2 + 1\right). \tag{27}$$

6. Holomorphic disks

We prove the following in this section.

Proposition 26. *The pair* $(\mathbb{CP}^3, \hat{P})$ *corresponding to a function* $h(t) \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$ *satisfies the condition* $(\sharp)$.

We first study the holomorphic disks which do not contain the singular point y_0. Recall that any holomorphic disk in $(\mathbb{CP}^2, \mathbb{RP}^2)$ whose relative homology class generates $H_2(\mathbb{CP}^2, \mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}$ is given by one of the hemispheres of a complex line (pp. 498–500 of [10]). We call such a disk the *standard disk* in $(\mathbb{CP}^2, \mathbb{RP}^2)$. Note that $\hat{P}$ is homotopic to the standard $\mathbb{RP}^3$ in $\mathbb{CP}^3$, so $H_2(\mathbb{CP}^3, \hat{P}; \mathbb{Z}) \simeq \mathbb{Z}$.

Lemma 27. *Let D be the complex unit disk and $\varphi : D \to \mathcal{O}(1)$ be a continuous map with $\varphi(\partial D) \subset P$. If φ is holomorphic on the interior of D and the relative homology class $[\varphi]$ generates $H_2(\mathbb{CP}^3, \hat{P}; \mathbb{Z})$, then φ is a holomorphic lift of some standard disk in $(\mathbb{CP}^2, \mathbb{RP}^2)$, i.e. the image of the composition $\pi \circ \varphi$ is a standard disk where $\pi : \mathcal{O}(1) \to \mathbb{CP}^2$ is the projection.*

Proof. Let $i : \mathcal{O}(1) \to \mathbb{CP}^3$ be the inclusion, then we have the isomorphisms

$$\mathbb{Z} \cong H_2(\mathbb{CP}^2, \mathbb{RP}^2) \xleftarrow{\pi_*} H_2(\mathcal{O}(1), P) \xrightarrow{i_*} H_2(\mathbb{CP}^3, \hat{P}) \cong \mathbb{Z}.$$

Since $[i \circ \varphi]$ generates $H_2(\mathbb{CP}^3, \hat{P})$, $[\pi \circ \varphi]$ also generates $H_2(\mathbb{CP}^2, \mathbb{RP}^2)$. Because $\pi \circ \varphi$ defines a holomorphic disk in $(\mathbb{CP}^2, \mathbb{RP}^2)$, it is a standard disk. $\square$

Now let D be a standard disk in $(\mathbb{CP}^2, \mathbb{RP}^2)$ and $\mathcal{F}_D$ be the set of holomorphic lifts of D in $(\mathcal{O}(1), P)$. In the proof of the next lemma, we follow the method of LeBrun and Mason [10,11].

Lemma 28. *$\mathcal{F}_D$ has a structure of a smooth family of holomorphic disks parameterized by $\mathbb{R}^2$.*

Remark 29. In our situation, we can calculate explicitly, and we have no need to use the inverse function theorem of Banach space. Therefore, we can treat many metrics which are far from the standard one.

Proof (*Proof of* 28). Recall that $\mathcal{O}(1) \cong \mathbb{CP}^3 - \{y_0\}$ ($y_0 = [0 : 0 : 0 : 1]$) and the projection $\pi : \mathcal{O}(1) \to \mathbb{CP}^2$ is given by $[z_1 : z_2 : z_3 : z_4] \mapsto [z_1 : z_2 : z_3]$. Let U be the affine open set in $\mathcal{O}(1)$ whose coordinate is defined by

$$\mathfrak{z}_1 = \frac{z_2 - \mathrm{i}z_1}{z_2 + \mathrm{i}z_1}, \qquad \mathfrak{z}_2 = \frac{z_3}{z_2 + \mathrm{i}z_1}, \qquad \mathfrak{z}_3 = \frac{z_4}{z_2 + \mathrm{i}z_1}.$$

Then the intersection $B := \mathcal{O}_{\mathbb{R}}(1) \cap U$ is equal to the set given by

$$\mathfrak{z}_1\overline{\mathfrak{z}_1} = 1, \qquad \mathfrak{z}_1\overline{\mathfrak{z}_2} = \mathfrak{z}_2, \qquad \mathfrak{z}_1\overline{\mathfrak{z}_3} = \mathfrak{z}_3.$$

We can parameterize B by $(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3) = (\mathrm{e}^{\mathrm{i}\theta}, t_1\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}, t_2\mathrm{e}^{\frac{\mathrm{i}\theta}{2}})$ by using θ, t_1 and t_2, or equivalently $z = [-\sin\frac{\theta}{2} : \cos\frac{\theta}{2} : t_1 : t_2]$. Therefore, we have $B \cong \mathbb{R}^3/\mathbb{Z}$ where the $\mathbb{Z}$-action is generated by $(\theta, t_1, t_2) \mapsto (\theta + 2\pi, -t_1, -t_2)$. Similarly, we can parameterize $B' := P \cap U$ by

$$(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3) = (\mathrm{e}^{\mathrm{i}\theta}, t_1\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}, [t_2 + h(t_1)]\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}).$$

Notice that $(\mathfrak{z}_1, \mathfrak{z}_2)$ defines a coordinate on the open set $\pi(U)$ in $\mathbb{CP}^2$, and that $\pi(B') = \pi(B) \cong \mathbb{R}^2/\mathbb{Z}$ is equal to $\mathbb{RP}^2 - \{[0 : 0 : 1]\}$.

The boundary ∂D of a standard holomorphic disk D is a real projective line in $\mathbb{RP}^2$. Each real line in $\mathbb{RP}^2$ is described by either form of

(1) $(\mathfrak{z}_1, \mathfrak{z}_2) = (\mathrm{e}^{\mathrm{i}\theta}, a + \overline{a}\mathrm{e}^{\mathrm{i}\theta})$ for some $a \in \mathbb{C}$,
(2) $(\mathfrak{z}_1, \mathfrak{z}_2) = (\mathrm{e}^{2\mathrm{i}\alpha}, \xi\mathrm{e}^{\mathrm{i}\alpha})$ for some $\alpha \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$,

where $\theta \in S^1$ and $\xi \in \mathbb{R}$ are the parameters. The case (2) occurs when the circle passes through $[0 : 0 : 1]$. Notice that in the case (2), α and $\alpha + \pi$ correspond to the same line with opposite orientations.

We give the proof for the two cases separately; one is the case when ∂D is given by (1) and the other case is by (2). In case (1), any lift of ∂D is described by

$$(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3) = \left(\mathrm{e}^{\mathrm{i}\theta}, a + \overline{a}\mathrm{e}^{\mathrm{i}\theta}, \left[u(\theta) + h\left(a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}} + \overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right)\right]\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right) \tag{28}$$

with the parameter $\theta \in S^1$, where $u(\theta)$ is an unknown real function satisfying $u(\theta + 2\pi) = -u(\theta)$. Notice that for a fixed $a \in \mathbb{C}$, there are two possibilities of D, i.e. the upper and lower hemispheres. Corresponding to this, we want to choose $u(\theta)$ so that the circle (28) extends holomorphically to the interior or exterior region of $\{|\omega| = 1\}$ where $\omega = \mathrm{e}^{\mathrm{i}\theta}$. In the interior case, we want to choose $u(\theta)$ such that $\mathfrak{z}_3$ contains no negative power in its Fourier expansion. Because u is real valued and h is pure imaginary valued, we can expand

$$u(\theta) = \sum_{l=0}^{\infty}\left\{u_l\,\mathrm{e}^{\mathrm{i}\theta(l+\frac{1}{2})} + \overline{u}_l\,\mathrm{e}^{-\mathrm{i}\theta(l+\frac{1}{2})}\right\}, \tag{29}$$

$$h\left(a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}} + \overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right) = \sum_{l=0}^{\infty}\left\{h_{a,l}\,\mathrm{e}^{\mathrm{i}\theta(l+\frac{1}{2})} - \overline{h}_{a,l}\,\mathrm{e}^{-\mathrm{i}\theta(l+\frac{1}{2})}\right\}. \tag{30}$$

Therefore, if we put $u_l = h_{a,l}$ for $l \geq 1$, then we have

$$\mathfrak{z}_3 = \left[u(\theta) + h\left(a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}} + \overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right)\right]\mathrm{e}^{\frac{\mathrm{i}\theta}{2}} = 2\sum_{l=0}^{\infty} h_{a,l}\,\mathrm{e}^{\mathrm{i}\theta(l+1)} + \kappa + \overline{\kappa}\mathrm{e}^{\mathrm{i}\theta},$$

where $\kappa = \overline{u}_0 - \overline{h}_{a,l}$ can be taken as an arbitrary complex constant. The circle (28) is described in the homogeneous coordinate by

$$[1 : \zeta : -2(\operatorname{Im} a) + 2(\operatorname{Re} a)\zeta : -2(\operatorname{Im}\kappa) + 2(\operatorname{Re}\kappa)\zeta + F(a, \omega(\zeta))] \tag{31}$$

with the parameter $\zeta \in \mathbb{R} \cup \{\infty\}$, where

$$\omega(\zeta) = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}, \qquad F(a, \omega) = \frac{4\mathrm{i}}{1 - \omega}\sum_{l=0}^{\infty} h_{a,l}\,\omega^{l+1}. \tag{32}$$

The circle (31) extends holomorphically to $\{\operatorname{Im}\zeta \geq 0\}$ and $\{|\omega| \leq 1\}$, and hence $\mathcal{F}_D$ is a smooth family parameterized by $\kappa \in \mathbb{C} \simeq \mathbb{R}^2$.

The exterior case is, in the same way, given by (31) using $\overline{F(a, \overline{\omega}^{-1})}$ instead of $F(a, \omega)$. In this case, (31) extends to $\{\operatorname{Im}\zeta \leq 0\}$ and $\{|\omega| \geq 1\}$.

In case (2), it is enough to consider the standard holomorphic disk given by $\{\xi \in \mathbb{C} : \operatorname{Im}\xi \geq 0\}$. In this case, any lift of ∂D is described by

$$(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3) = (\mathrm{e}^{2\mathrm{i}\alpha}, \xi\mathrm{e}^{\mathrm{i}\alpha}, [u(\xi) + h(\xi)]\mathrm{e}^{\mathrm{i}\alpha}) \tag{33}$$

with the parameter $\xi \in \mathbb{R}$, where $u(\xi)$ is an unknown real function. We define a holomorphic function $G(\xi)$ on $\{\operatorname{Im}\xi > 0\}$ by

$$G(\xi) = \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{h(\mu)}{\mu-\xi}\mathrm{d}\mu.$$

Then $G(\xi)$ extends smoothly to $\{\operatorname{Im}\xi = 0\}$ by

$$G(\xi) = \frac{1}{\pi \mathrm{i}}\,\mathbf{pv.}\int_{-\infty}^{\infty}\frac{h(\mu)}{\mu-\xi}\mathrm{d}\mu + h(\xi). \tag{34}$$

The first term of the right-hand side is equal to $-\mathcal{H}h(\xi)$ where $\mathcal{H}$ is the Hilbert transform defined by (A.4). If we put $v(\xi) = u(\xi) + \mathcal{H}h(\xi)$, then we have

$$u(\xi) + h(\xi) = v(\xi) + G(\xi)$$

on $\{\operatorname{Im}\xi = 0\}$. The circle (33) is described in the homogeneous coordinate by

$$[-\sin\alpha : \cos\alpha : \xi : v(\xi) + G(\xi)] \tag{35}$$

with the parameter $\xi \in \mathbb{R}$. If the circle (35) is bounded by a holomorphic disk, $v(\xi)$ should extend holomorphically to the upper half plane of ξ. Because $v(\xi)$ is real valued on $\mathbb{R}$, it extends to a holomorphic function on whole $\mathbb{C}$. Hence $v(\xi)$ expands to a non-negative power series of ξ with real coefficients.

Moreover, (35) converges to $y_0 = [0 : 0 : 0 : 1]$ by $\xi \to \infty$, if and only if $v(\xi)$ has any term higher than or equal to ξ^2. These cases are removable, because the disk is not contained in $\mathcal{O}(1)$. Hence $v(\xi)$ should be a degree-one polynomial. Because the ambiguity is given by the coefficients of $v(\xi)$, $\mathcal{F}_D$ is a smooth $\mathbb{R}^2$-family of holomorphic disks. □

Let $\mathcal{F}$ be the set of all the holomorphic disks in $(\mathcal{O}(1), P)$ such that the relative homology class of each disk generates $H_2(\mathbb{CP}^3, \hat{P}; \mathbb{Z})$. Then we have $\mathcal{F} = \cup\mathcal{F}_D$ from Lemma 27.

Lemma 30. *$\mathcal{F}$ has a structure of a smooth family of holomorphic disks parameterized by TS^2 such that interiors of the disks of $\mathcal{F}$ foliate $\mathcal{O}(1) - \mathcal{O}(1)\,|_{\mathbb{RP}^2}$.*

Proof. Recall the diagram (5), and we construct a smooth map $\Phi_{\mathbb{C}} : \mathcal{Z} \to \mathcal{O}(1)$ as a deformation of $\Phi_{\mathbb{C},0}$ such that $\{\Phi_{\mathbb{C}}(\tilde{\mathfrak{p}}^{-1}(x))\}_{x\in TS^2}$ is equal to $\mathcal{F}$. Using the coordinate defined in Section 2, we define $\Phi_{\mathbb{C}}$ on $\mathcal{Z}\,|_{D_+}$ by

$$\begin{aligned}&\left(x_1^+, x_2^+, x_3^+, x_4^+, \zeta^+\right) \longmapsto \left[1 : \zeta^+ : -x_1^+ - x_2^+\zeta^+ : -x_3^+\zeta^+ + x_4^+ + H^+(x_1^+, x_2^+, \zeta^+)\right],\\ &H^+(x_1^+, x_2^+, \zeta^+) = F\left(a, \omega(\zeta^+)\right), \quad a = \frac{\mathrm{i}}{2}(x_1^+ + \mathrm{i}x_2^+),\end{aligned} \tag{36}$$

where F is given in (32). Similarly, we define $\Phi_{\mathbb{C}}$ on $\mathcal{Z}\,|_{D_-}$ by the same formula as (36) by exchanging the sign $\cdot^+$ with $\cdot^-$ and

$$H^-(x_1^-, x_2^-, \zeta^-) = -\overline{F\left(a, \overline{\omega(\zeta^-)}^{-1}\right)}, \qquad a = \frac{\mathrm{i}}{2}(x_1^- + \mathrm{i}x_2^-).$$

Note that the parameters run $\{\operatorname{Im}\zeta^+ \geq 0\}$ and $\{\operatorname{Im}\zeta^- \leq 0\}$. On the other hand, we define $\Phi_{\mathbb{C}}$ on $\mathcal{Z}\,|_{W_0} = \{\beta = 0\}$ by

$$(\alpha, 0, \varepsilon_1, \varepsilon_2, \xi) \longmapsto [-\sin\alpha : \cos\alpha : \xi : -\varepsilon_1\xi + \varepsilon_2 + G(\xi)]\,. \tag{37}$$

From (31) and (35), we obtain that $\{\Phi_{\mathbb{C}}(\tilde{\mathfrak{p}}^{-1}(x))\}_{x\in TS^2}$ is equal to $\mathcal{F}$. Hence $\mathcal{F}$ is parameterized by TS^2.

We have to prove that the above $\Phi_{\mathbb{C}}$ is smooth. We now check that (36) and (37) are continued. Here we omit the sign '+' for the simplicity. Using the coordinate change

$$\xi = -\mathrm{i}\cot\beta\,\frac{\omega - \mathrm{e}^{2\mathrm{i}\alpha}}{\omega + \mathrm{e}^{2\mathrm{i}\alpha}},$$

which is obtained from (3) and (32), and so on, we have

$$\begin{aligned}&[1 : \zeta : -x_1 - x_2\zeta : -x_3\zeta + x_4 + H(x_1, x_2, \zeta)]\\ &\quad = [\xi\cos\alpha\tan\beta - \sin\alpha : -\xi\sin\alpha\tan\beta + \cos\alpha : \xi : -\varepsilon_1\xi + \varepsilon_2 + B(\alpha, \beta, \xi)]\,,\end{aligned}$$

where

$$B(\alpha,\beta,\xi)=\frac{4\mathrm{e}^{\mathrm{i}\alpha}}{\omega+\mathrm{e}^{2\mathrm{i}\alpha}}\sum_{l=0}^{\infty}h_{a,l}\,\omega^{l+1}. \tag{38}$$

In general, we define the operator Π on the L^2-functions on $S^1=\{|\omega|=1\}$ by

$$\Pi : u(\omega)=\sum_{k=-\infty}^{\infty}u_k\omega^k \longmapsto \sum_{k=0}^{\infty}u_k\omega^k.$$

As explained in [15], when $|\omega|<1$ we have

$$\Pi u(\omega)=\frac{1}{2\pi\mathrm{i}}\int_{S^1}\frac{u(\eta)}{\eta-\omega}\mathrm{d}\eta=\frac{1}{2\pi}\int_{S^1}\frac{u(\mathrm{e}^{\mathrm{i}\theta})}{\mathrm{e}^{\mathrm{i}\theta}-\omega}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

Hence, in our case,

$$\begin{aligned}\sum_{l=0}^{\infty}h_{a,l}\,\omega^l &= \Pi\left(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}h\left(a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}+\overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right)\right)(\omega)\\ &=\frac{1}{2\pi}\int_{S^1}\frac{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}h\left(a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}+\overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right)}{\mathrm{e}^{\mathrm{i}\theta}-\omega}\mathrm{d}\theta.\end{aligned}$$

If we change the parameter θ by $\mu=a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}+\overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}=\cot\beta\sin(\frac{\theta}{2}-\alpha)$, then we have

$$\begin{aligned}B(\alpha,\beta,\xi)&=\frac{2\mathrm{e}^{\mathrm{i}\alpha}\omega}{\pi(\omega+\mathrm{e}^{2\mathrm{i}\alpha})}\int_{-\cot\beta}^{\cot\beta}\frac{h(\mu)}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}-\omega\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}\cdot\frac{2\mathrm{d}\mu}{\cot\beta\,\cos(\frac{\theta}{2}-\alpha)}\\ &=\frac{1+\xi^2\tan^2\beta}{\pi\mathrm{i}}\int_{-\cot\beta}^{\cot\beta}\frac{h(\mu)}{\mu-\xi\sqrt{1-\mu^2\tan^2\beta}}\cdot\frac{\mathrm{d}\mu}{\sqrt{1-\mu^2\tan^2\beta}}.\end{aligned}$$

Because $h(t)$ is rapidly decreasing, $B(\alpha,\beta,\xi)$ extends continuously to $\beta=0$ and we have $B(\alpha,0,\xi)=G(\xi)$. Hence $\Phi_{\mathbb{C}}$ is continuous on $\mathcal{Z}\,|_{D_+\cup W_0}$. In the same vein, $\Phi_{\mathbb{C}}$ is continuous on $\mathcal{Z}\,|_{D_-\cup W_0}$. Moreover, it is smooth because of the above formula.

We check that $\mathcal{O}(1)-\mathcal{O}(1)\,|_{\mathbb{RP}^2}$ is foliated by $\mathcal{F}$. For any $u\in\mathcal{O}(1)-\mathcal{O}(1)\,|_{\mathbb{RP}^2}$, there is a unique standard disk D in $(\mathbb{CP}^2,\mathbb{RP}^2)$ which contains $\pi(u)\in\mathbb{CP}^2-\mathbb{RP}^2$. Then, from (31) or (35), there is a unique holomorphic disk in $\mathcal{F}_D$ which contains u. □

Remark 31. The smoothness of $\Phi_{\mathbb{C}}$ is also checked in the proof of Lemma 33, actually $\Phi_{\mathbb{C}}$ is holomorphic on the interior of $\mathcal{Z}$ for some complex structure on there.

Remark 32. For each $t\in S^2$, the set of holomorphic disks $\{\Phi_{\mathbb{C}}(\tilde{\mathfrak{p}}^{-1}(x))\}_{x\in T_tS^2}$ is equal to $\mathcal{F}_D$, where $D=\mathfrak{q}(\mathfrak{p}^{-1}(t))$ is the standard disk corresponding to t in the twistor correspondence for the standard Zoll projective structure.

Proof (*Proof of* 26). Notice that $\mathbb{CP}^3-\hat{P}=(\mathcal{O}(1)-\mathcal{O}(1)\,|_{\mathbb{RP}^2})\coprod(\mathcal{O}(1)\,|_{\mathbb{RP}^2}-P)$. We require a family of holomorphic disks through y_0 whose interiors foliate $\mathcal{O}(1)\,|_{\mathbb{RP}^2}-P$. Similarly to the standard case, such a family is given by the disks

$$\{[z_1:z_2:z_3:u+h(z_3)] : \mathrm{Im}\,u\geq 0\}\cup\{y_0\} \tag{39}$$

for some $(z_1,z_2,z_3)\in\mathbb{R}^3-\{0\}$. There is a one-to-one correspondence between S^2_∞ and the above holomorphic disks which is given by putting $(z_1,z_2,z_3)=t\times v$ for each point in S^2_∞ of the form (7).

If we define $\hat{\mathcal{F}}$ as the union of $\mathcal{F}$ and the disks of the form (39), then $\hat{\mathcal{F}}$ is a continuous family of holomorphic disks in $(\mathbb{CP}^3,\hat{P})$ parameterized by $\widetilde{Gr}_2(\mathbb{R}^4)$. The conditions ($\sharp$ 1), ($\sharp$ 2), and the uniqueness for $\hat{\mathcal{F}}$ follow directly from the construction. □

7. Twistor correspondence

In this section, we give the proof of Theorem 24.

Lemma 33. *For a given twistor space $(\mathcal{O}(1), P)$ corresponding to a function $h \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$, there are a smooth map $\Phi_{\mathbb{C}} : \mathcal{Z} \to \mathcal{O}(1)$ and a function $f \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ which satisfy the following properties:*

(1) $\Phi_{\mathbb{C},x} : \mathcal{Z}_x \to \mathcal{O}(1)$ *is holomorphic on the interior of* $\mathcal{Z}_x = \tilde{\mathfrak{p}}^{-1}(x)$ *and* $\Phi_{\mathbb{C},x}(\partial\mathcal{Z}_x) \subset P$*, where* $\Phi_{\mathbb{C},x}$ *is the restriction of* $\Phi_{\mathbb{C}}$ *on* $\mathcal{Z}_x$*,*
(2) $\Phi_{\mathbb{C}}$ *is injective on* $\mathcal{Z} - \partial\mathcal{Z}$*,*
(3) $\{\tilde{\mathfrak{p}}(\Phi_{\mathbb{C}}^{-1}(y))\}_{y\in P}$ *is equal to the set of* β*-surfaces on* TS^2*,*

respecting the self-dual metric on TS^2 and the complex structure on $\mathcal{Z}$ corresponding to f. Such f is given by

$$f(x) = 2\mathrm{i}\left(\frac{\partial h}{\partial t}\right)^{\vee}(x). \tag{40}$$

Proof. The map $\Phi_{\mathbb{C}}$ is already constructed in Lemma 30, and the conditions (1) and (2) are already checked. We now construct $f \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ and check the condition (3). If $\Phi_{\mathbb{C}}$ is holomorphic on the interior of $\mathcal{Z}$ with respect to the complex structure defined from f, then Eq. (27) holds, and this is equivalent to

$$\left(\omega\frac{\partial}{\partial a} - \frac{\partial}{\partial \overline{a}}\right)((1-\omega)F(a,\omega)) = -2\omega f. \tag{41}$$

For given h, the function f is uniquely defined by (41). Actually, from the identity

$$\left(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\frac{\partial}{\partial a} - \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}\frac{\partial}{\partial \overline{a}}\right) h\left(a\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}} + \overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right) = 0,$$

we obtain

$$\frac{\partial h_{a,l}}{\partial \overline{a}} = \frac{\partial h_{a,l-1}}{\partial a} \quad l \geq 1, \quad \text{and} \quad \frac{\partial h_{a,0}}{\partial \overline{a}} \text{ is real valued.}$$

Therefore, Eq. (41) holds if and only if $f(x) = 2\mathrm{i}\frac{\partial h_{a,0}}{\partial \overline{a}}$, where $x = (x_1^+, x_2^+)$ and $a = \frac{\mathrm{i}}{2}(x_1^+ + \mathrm{i}x_2^+)$. Here we have

$$\begin{aligned}\frac{\partial h_{a,0}}{\partial \overline{a}} &= \frac{1}{2\pi}\frac{\partial}{\partial \overline{a}}\int_{S^1} \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}} h\left(a\mathrm{e}^{\frac{-\mathrm{i}\theta}{2}} + \overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right)\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{S^1} \frac{\partial h}{\partial t}\left(a\mathrm{e}^{\frac{-\mathrm{i}\theta}{2}} + \overline{a}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\right)\mathrm{d}\theta\\ &= \left(\frac{\partial h}{\partial t}\right)^{\vee}(x),\end{aligned}$$

hence we put

$$f(x) = 2\mathrm{i}\left(\frac{\partial h}{\partial t}\right)^{\vee}(x) \tag{42}$$

which is real valued, rapidly decreasing and axisymmetric.

Now we prove that condition (3) holds for this f. Eq. (27) holds on $\{\zeta^+ \in \mathbb{C} : \mathrm{Im}\,\zeta^+ = 0\}$, therefore, for a fixed $\zeta^+ \in \mathbb{R}$, the functions

$$-x_1^+ - x_2^+\zeta^+ \quad \text{and} \quad -x_3^+\zeta^+ + x_4^+ + H(x_1^+, x_2^+, \zeta^+)$$

are annihilated by $\mathfrak{n}_1, \mathfrak{n}_2$ given by (13). Hence, $\tilde{\mathfrak{p}}(\Phi^{-1}(y))$ is a β-surface on $TS^2|_{D_+}$. In the same way, $\tilde{\mathfrak{p}}(\Phi^{-1}(y))$ is a β-surface on $TS^2|_{D_-}$ therefore, $\tilde{\mathfrak{p}}(\Phi^{-1}(y))$ is a β-surface on TS^2. Hence, condition (3) holds. □

Lemma 34. *For a given self-dual metric on TS^2 corresponding to a function $f \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$, there are a function $h \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$ and a smooth surjection $\Phi_{\mathbb{C}} : \mathcal{Z} \to \mathcal{O}(1)$ which satisfy conditions* (1), (2) *and* (3) *in Lemma 33 respecting the complex structure on $\mathcal{Z}$ defined from the self-dual metric and the twistor space $(\mathcal{O}(1), P)$ corresponding to h. Such h is given by*

$$h(t) = -\frac{1}{4}\mathcal{H}\hat{f}(t). \tag{43}$$

Proof. We have already seen in (26) that there is a natural map $\Phi : F = \partial\mathcal{Z} \to \mathcal{O}_{\mathbb{R}}(1)$, which is given by

$$(\alpha, 0, \varepsilon_1, \varepsilon_2, \xi) \longmapsto \left[-\sin\alpha : \cos\alpha : \xi : -\varepsilon_1\xi + \varepsilon_2 + \frac{1}{4}\hat{f}(\xi)\right], \tag{44}$$

on $F|_{W_0}$. Now we deform this map so as to extend holomorphically to the upper half plane of ξ. For this purpose, we put

$$h(\xi) = \frac{1}{4\pi \mathrm{i}}\mathbf{pv.}\int_{-\infty}^{\infty}\frac{\hat{f}(\mu)}{\mu-\xi}\mathrm{d}\mu = -\frac{1}{4}\mathcal{H}\hat{f}(\xi) \tag{45}$$

which is odd, rapidly decreasing and pure imaginary valued. As in the previous section, $G(\xi) = h(\xi) + \frac{1}{4}\hat{f}(\xi)$ extends holomorphically to $\{\mathrm{Im}\,\xi > 0\}$ by

$$G(\xi) = \frac{1}{4\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{\hat{f}(\mu)}{\mu-\xi}\mathrm{d}\mu.$$

Instead of (44), we define a map $\Upsilon : \mathcal{Z}|_{W_0} \to \mathcal{O}(1)$ by

$$(\alpha, 0, \varepsilon_1, \varepsilon_2, \xi) \longmapsto [-\sin\alpha : \cos\alpha : \xi : -\varepsilon_1\xi + \varepsilon_2 + G(\xi)]\,.$$

Let $(\mathcal{O}(1), P)$ be the twistor space corresponding to $h \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$, then $\Upsilon_x : \mathcal{Z}_x \to \mathcal{O}(1)$ is a holomorphic disk in $\mathcal{O}(1)$ with boundary on P for each $x \in TS^2|_{W_0}$.

By Lemma 33, we can construct a smooth map $\Phi_{\mathbb{C}} : \mathcal{Z} \to \mathcal{O}(1)$ with $\Phi_{\mathbb{C}}(\partial\mathcal{Z}) = P$, which is holomorphic on the interior of $\mathcal{Z}$ with respect to the complex structure defined from the function

$$k(x) = 2\mathrm{i}\left(\frac{\partial h}{\partial t}\right)^{\vee}(x).$$

Connecting with (45), we have

$$k(x) = \frac{1}{2\mathrm{i}}\left(\frac{\partial}{\partial t}\mathcal{H}\hat{f}\right)^{\vee}(x).$$

By the inversion formula (A.3), we have $k(x) = f(x)$. Therefore $\Phi_{\mathbb{C}}$ is holomorphic with respect to the given complex structure on $\mathcal{Z}$. Note that the restriction of $\Phi_{\mathbb{C}}$ to $\mathcal{Z}|_{W_0}$ is equal to Υ. Conditions (1), (2) and (3) follow from Lemma 33. □

Lemma 35. (43) *defines a one-to-one correspondence between $f(x) \in \mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ and $h(\xi) \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$.*

Proof. From Schwartz's theorem (Theorem 2.4 of [7]), we can check that the Radon transform $f(x) \to \varphi(p) = \hat{f}(p)$ is one-to-one from $\mathcal{S}(\mathbb{R}^2)^{\mathrm{sym}}$ to $\mathcal{S}(\mathcal{P})^{\mathrm{sym}}$, where $\mathcal{P}$ is the set of unoriented lines in $\mathbb{R}^2$ and $\mathcal{S}(\mathcal{P})^{\mathrm{sym}}$ is the set of rapidly decreasing functions on $\mathcal{P}$ which depend only on the distance from the origin. Then we can identify $\mathcal{S}(\mathcal{P})^{\mathrm{sym}}$ with $\mathcal{S}(\mathbb{R})^{\mathrm{even}}$ the set of rapidly decreasing even functions on $\mathbb{R}$.

On the other hand, the Hilbert transform $\varphi \to \mathcal{H}\varphi$ is involutive, i.e. $\mathcal{H}^2 = \mathrm{id}$. Moreover, $\mathcal{H}$ exchanges odd functions and even functions, or real valued functions and pure imaginary valued functions. Altogether, the statement holds. □

Proof (*Proof of* 24). The one-to-one correspondence is already given in Lemmas 33–35. Therefore, we construct the double fibration $\widetilde{Gr}_2(\mathbb{R}^4) \leftarrow \hat{\mathcal{Z}} \to \mathbb{CP}^3$ and we check conditions (1), (2), (3) and (4). As explained in the proof of

Proposition 26, the family of holomorphic disks in $(\mathbb{CP}^3, \hat{P})$ is parameterized by $\widetilde{Gr}_2(\mathbb{R}^4)$ for each $h \in \mathrm{i}\mathcal{S}(\mathbb{R})^{\mathrm{odd}}$. Let D_x be the holomorphic disk in $(\mathbb{CP}^3, P)$ which corresponds to $x \in \widetilde{Gr}_2(\mathbb{R}^4)$, and we put

$$\hat{\mathcal{Z}} = \left\{(x, y) \in \widetilde{Gr}_2(\mathbb{R}^4) \times \mathbb{CP}^3 : y \in D_x\right\}.$$

We define $\wp : \hat{\mathcal{Z}} \to \widetilde{Gr}_2(\mathbb{R}^4)$ and $\Psi : \hat{\mathcal{Z}} \to \mathbb{CP}^3$ as the projections. Respecting the natural embedding $\mathcal{Z} \to \hat{\mathcal{Z}}$, $\wp$ and Ψ are natural extension of $\tilde{\mathfrak{p}}$ and $\Phi_{\mathbb{C}}$, respectively. From Proposition 26 and its proof, $\wp$ is a disk bundle with fiberwise complex structure, and Ψ is a continuous surjection, so the condition (1) holds. Conditions (2) and (3) follow from Proposition 26, Lemmas 33 and 34.

Lastly, condition (4) follows from the continuity of the family of holomorphic disks. Actually if $y \neq y_0$ then $\wp(\Psi^{-1}(y))$ is the closure of the β-surface $\tilde{\mathfrak{p}}(\Phi_{\mathbb{C}}^{-1}(y)) \subset TS^2$, and if $y = y_0$ then $\wp(\Psi^{-1}(y_0)) = S^2_\infty$ is the singular β-surface. $\square$

From condition (3) of Theorem 24, $\wp$ induces a continuous map $\varpi : (\mathbb{CP}^3 - \hat{P}) \to \widetilde{Gr}_2(\mathbb{R}^4)$ which is smooth on $\mathcal{O}(1) - \mathcal{O}(1)|_{\mathbb{RP}^2}$. The next proposition says that ϖ is not differentiable on $\mathcal{O}(1)|_{\mathbb{RP}^2} - P$ when the singularity exists.

Proposition 36. *ϖ is differentiable if and only if $h(\mu) \equiv 0$.*

Proof. Let $V = \{[1 : w_1 : w_2 : w_3] \in \mathbb{CP}^3\}$ be an affine open set. Then $V \cap \hat{P}$ is given by $\{(w_1, w_2, w_3) : \operatorname{Im} w_1 = \operatorname{Im} w_2 = \operatorname{Im} w_3 - \operatorname{Im} H = 0\}$, where H is a function of $\{(w_1, w_2) : \operatorname{Im} w_1 \neq 0\}$ given by

$$H = H^{\pm}\left(\frac{\operatorname{Im} \overline{w}_1 w_2}{\operatorname{Im} w_1}, -\frac{\operatorname{Im} w_2}{\operatorname{Im} w_1}, w_1\right)$$

by using $H^{\pm}$ defined in (36) and so on, and $\operatorname{Im} H$ extends continuously on $\{\operatorname{Im} w_1 = 0\}$ from the definition. Let U be an open neighborhood of $\widetilde{Gr}_2(\mathbb{R}^4)$ given in the proof of Proposition 17. Then the image $\varpi(V \setminus \hat{P})$ is contained in U and $\varpi|_{V\setminus\hat{P}}$ is described by

$$u_1 = \frac{\operatorname{Im} \overline{w}_1(w_3 - H)}{\operatorname{Im}(w_3 - H)}, \qquad u_2 = -\frac{\operatorname{Im} w_1}{\operatorname{Im}(w_3 - H)},$$
$$u_3 = -\frac{\operatorname{Im} w_2}{\operatorname{Im}(w_3 - H)}, \qquad u_4 = \frac{\operatorname{Im} \overline{w}_2(w_3 - H)}{\operatorname{Im}(w_3 - H)}.$$

Note that these equations are defined on $\{\operatorname{Im} w_1 \neq 0\} \cap (V \setminus \hat{P})$; however, these are continuously extended on $V \setminus \hat{P}$.

Now we prove that if ϖ is differentiable, then, for all $A > 0$, $h(\mu) = 0$ for any $\mu \in [-A, A]$. Let $t \in (-\varepsilon, \varepsilon)$ be a parameter on a small interval, and we fix $s \in \mathbb{R}$. We take a smooth curve in $V \setminus \hat{P}$ given by

$$(w_1, w_2, w_3) = (s + \mathrm{i}t, A(-s + \mathrm{i}t), c(t)),$$

where $c(t)$ is a smooth function. Then

$$H = H(t) = H^{\pm}(0, -A, s + \mathrm{i}t)$$

is defined on $t \neq 0$ and $\operatorname{Im} H(t)$ is defined on $t \in (-\varepsilon, \varepsilon)$. We have $\operatorname{Im} c(t) - \operatorname{Im} H(t) \neq 0$ for all $t \in (-\varepsilon, \varepsilon)$, and

$$u_4 = u_4(t) = -\frac{\operatorname{Re}(c(t) - H(t))}{\operatorname{Im}(c(t) - H(t))}\, t - s.$$

Hence

$$\lim_{t\to 0} \frac{\mathrm{d}u_4}{\mathrm{d}t} = -\lim_{t\to 0} \frac{\operatorname{Re}(c(t) - H(t))}{\operatorname{Im}(c(t) - H(t))}.$$

Since $\operatorname{Im} H(t)$ is continuous, $\operatorname{Re} H(t)$ is also continuous if ϖ is differentiable. By definition, we have

$$H(t) = H^{+}(0, -A, s + \mathrm{i}t) = F\left(\frac{A}{2}, \omega\right) = \frac{4\mathrm{i}}{1 - \omega} \sum_{l=0}^{\infty} h_{\frac{A}{2},l}\, \omega^{l+1} \quad \text{on } t > 0,$$

$$H(t) = H^{-}(0, -A, s + \mathrm{i}t) = -\overline{F\left(\frac{A}{2}, \overline{w}^{-1}\right)} = -\frac{4\mathrm{i}}{1 - \omega} \sum_{l=0}^{\infty} \overline{h_{\frac{A}{2},l}}\, \omega^{-l} \quad \text{on } t < 0,$$

where $\omega = \omega(t) = \frac{s+\mathrm{i}t-\mathrm{i}}{s+\mathrm{i}t+\mathrm{i}}$. If we evaluate $a = \frac{A}{2}$ to the expansion of h given in (30), we have

$$h\left(A\cos\frac{\theta}{2}\right) = \sum_{l=0}^{\infty}\left\{h_{\frac{A}{2},l}\mathrm{e}^{\mathrm{i}\theta(l+\frac{1}{2})} - \overline{h_{\frac{A}{2},l}}\mathrm{e}^{-\mathrm{i}\theta(l+\frac{1}{2})}\right\}. \tag{46}$$

Then we have

$$\lim_{t\searrow 0}\operatorname{Im} H(t) = \lim_{t\nearrow 0}\operatorname{Im} H(t) = \frac{\mathrm{i}}{\sin\frac{\theta}{2}}\, h\left(A\cos\frac{\theta}{2}\right),$$

$$\lim_{t\searrow 0}\operatorname{Re} H(t) = -\lim_{t\nearrow 0}\operatorname{Re} H(t) = -\frac{1}{\sin\frac{\theta}{2}}\sum_{l=0}^{\infty}\left\{h_{\frac{A}{2},l}\mathrm{e}^{\mathrm{i}\theta(l+\frac{1}{2})} + \overline{h_{\frac{A}{2},l}}\mathrm{e}^{-\mathrm{i}\theta(l+\frac{1}{2})}\right\},$$

where we define $\theta = \theta(s) \in (0, 2\pi)$ so that $\mathrm{e}^{\mathrm{i}\theta} = \frac{s-\mathrm{i}}{s+\mathrm{i}} = \omega(0)$. The first equation states that $\operatorname{Im} H(t)$ is indeed continuous. The summand in the right-hand side of the second equation defines a Fourier expansion for some L^2-function for $\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}$ which is zero almost everywhere if $\operatorname{Re} H(t)$ is continuous for every $s \in \mathbb{R}$. Hence $h_{\frac{A}{2},l} = 0$ for all l, and we have $h(A\cos\frac{\theta}{2}) = 0$ for all θ from (46). Therefore, for all $A > 0$, $h(\mu) = 0$ for any $\mu \in [-A, A]$ if ϖ is differentiable. □

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Appendix. Radon transform

Here is a review of the Radon transform over $\mathbb{R}^2$ (cf. [7]). Let $\tilde{\mathcal{P}}$ be the set of oriented lines in $\mathbb{R}^2$. Then $\tilde{\mathcal{P}}$ is diffeomorphic to $S^1 \times \mathbb{R}$, where the correspondence $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \ni (\sigma, \mu) \mapsto \xi \in \tilde{\mathcal{P}}$ is given by $\xi = \{t(\cos\sigma, \sin\sigma) + \mu(-\sin\sigma, \cos\sigma)\}$ using parameter $t \in \mathbb{R}$. Let $\mathcal{P}$ be the set of unoriented lines in $\mathbb{R}^2$, then $\mathcal{P} \cong (S^1 \times \mathbb{R})/\mathbb{Z}_2$ where the equivalence is $(\sigma, \mu) \sim (\sigma + \pi, -\mu)$. Let f be a rapidly decreasing complex valued function over $\mathbb{R}^2$, then the *Radon transform* $\hat{f}$ of f is defined by

$$\hat{f}(\sigma, \mu) = \int_{-\infty}^{\infty} f(t\cos\sigma - \mu\sin\sigma, t\sin\sigma + \mu\cos\sigma)\mathrm{d}t. \tag{A.1}$$

Then $\hat{f}$ is a rapidly decreasing function of $\mathcal{P}$. The *dual transform* $\varphi^{\vee}$ of rapidly decreasing function φ of $\mathcal{P}$ is defined by

$$\varphi^{\vee}(x) = \frac{1}{2\pi}\int_{S^1}\varphi(\sigma, \mu(x, \sigma))\mathrm{d}\sigma, \tag{A.2}$$

where $\mu(x, \mu) = -x_1\sin\sigma + x_2\cos\sigma$. Note that $(\sigma, \mu(x, \sigma))$ runs all the oriented lines through $x \in \mathbb{R}^2$. The inversion formula is given by

$$f = \frac{1}{2\mathrm{i}}\left(\frac{\partial}{\partial\mu}\mathcal{H}_{\mu}\hat{f}\right)^{\vee}, \tag{A.3}$$

where $\mathcal{H}_{\mu}$ is the *Hilbert transform* defined by

$$(\mathcal{H}_{\mu}\varphi)(\sigma, \mu) = \frac{\mathrm{i}}{\pi}\,\mathbf{pv.}\int_{-\infty}^{\infty}\frac{\varphi(\sigma, \nu)}{\nu - \mu}\mathrm{d}\nu. \tag{A.4}$$

References

[1] D.E. Blair, A product twistor space, Serdica Math. J. 28 (2002) 163–174.
[2] D.E. Blair, J. Davidov, O. Muškarov, Hyperbolic twistor spaces, Rocky Mountain J. Math. 35 (2005) 1437–1466.
[3] D.M.J. Calderbank, Selfdual 4-manifolds, projective surfaces, and the Dunajski–West construction. math.DG/0606754, 2006. e-print.
[4] M. Dunajski, S. West, Anti-self-dual conformal structures with null Killing vectors from projective structures. math.DG/0601419, 2006. e-print.
[5] V. Guillemin, The Radon transform on Zoll surfaces, Adv. Math. 22 (1976) 85–119.
[6] V. Guillemin, Cosmology in (2 + 1)-dimensions, cyclic models, and deformations of $M_{2,1}$, in: Annals of Mathematics Studies, vol. 121, Princeton University Press, Princeton, 1989.
[7] S. Helgason, The Radon Transform, second edn., in: Progress in Mathematics, vol. 5, Birkhäusar, 1999.
[8] G. Jensen, M. Rigoli, Neutral surfaces in neutral four-spaces, Matematiche (Catania) 45 (1991) 407–443.
[9] H. Kamada, Compact scalar-flat indefinite Kähler surfaces with Hamiltonian S^1-symmetry, Comm. Math. Phys. 254 (2005) 23–44.
[10] C. LeBrun, L.J. Mason, Zoll manifolds and complex surfaces, J. Differential Geom. 61 (2002) 453–535.
[11] C. LeBrun, L.J. Mason, Nonlinear gravitons, null geodesics, and holomorphic disks. math.DG/0504582, 2005. e-print.
[12] Y. Matsushita, Four-dimensional Walker metric and symplectic structures, J. Geom. Phys. 52 (2004) 89–99.
[13] Y. Matsushita, Walker 4-manifolds with proper almost complex structures, J. Geom. Phys. 55 (2005) 385–398.
[14] J. Petean, Indefinite Kähler–Einstein metrics on compact complex surfaces, Comm. Math. Phys. 189 (1997) 227–235.
[15] M. Taylor, Pseudo differential operators, in: Lecture Notes in Mathematics, vol. 416, Springer-Verlag, Berlin, 1974.
[16] K.P. Tod, Indefinite conformally-ASD metric on $S^2 \times S^2$, in: Further Advances in Twistor Theory, vol. III, Chapman & Hall/CRC, 2001, pp. 61–63. Reprinted from Twistor Newsletter 36 (1993).